Appendix F

Elementary Concepts

AF.1 Coordinate Systems

We are interested in specifying the coordinates of the unit propagation vector $\hat{\Omega}$ in both the *Cartesian coordinate system* and the *spherical-polar coordinate system* (see Fig. 3.3). The spherical-polar system defines $\hat{\Omega}$ in terms of the two angles, θ and ϕ. The rectangular system defines $\hat{\Omega}$ in terms of its three projections in the (x, y, z) directions, Ω_x, Ω_y, and Ω_z. The relationships between these two sets of coordinates are

$$\Omega_x = \sin\theta\cos\phi; \qquad \Omega_y = \sin\theta\sin\phi; \qquad \Omega_z = \cos\theta \tag{F.1}$$

where $0 \le \phi \le 2\pi$, and $0 \le \theta \le \pi$.

AF.2 The Dirac Delta-function

A concept which is useful in the mathematical representation of unidirectional or *collimated light* is the *Dirac δ-function.* This 'function' has the peculiar property that it is zero for finite values of its argument, and unbounded (infinite) when the argument of the δ-function is zero, that is

$$\delta(x) = 0 \qquad (x \neq 0) \qquad \text{and} \qquad \delta(x) \to \infty \qquad (x \to 0). \tag{F.2}$$

Furthermore, the 'area' under the function is unity, that is, it is *normalized*

$$\begin{aligned} \int_a^b dx\delta(x) &= 1 \qquad \text{if a and b are of different sign.} \\ &= 0 \qquad \text{if a and b are of the same sign.} \end{aligned} \tag{F.3}$$

It is possible to define the δ-function for a *vector* argument. If we want to represent the electric field from a concentrated 'source' of unit strength (for example, an electron) at the point $\vec{r} = \vec{r}_0$, we write $\delta(\vec{r}-\vec{r}_0)$. In rectangular coordinates $\delta(\vec{r}-\vec{r}_0)$ can be defined as a product of one-dimensional δ-functions, that is

$$\delta(\vec{r}-\vec{r}_0) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0). \tag{F.4}$$

The integral properties analogous to those in eqn. (F.3) are

$$\int\int\int d^3\vec{r}\;\delta(\vec{r}-\vec{r}_0) = \int dx \int dy \int dz\;\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) = 1 \tag{F.5}$$

when the integration domain includes $\vec{r}_0$. The integral in eqn. (F.5) is zero if the integration domain does not include $\vec{r}_0$.

In spherical polar coordinates we represent

$$\delta(\vec{r}-\vec{r}_0) = \delta(\cos\theta - \cos\theta_0)\delta(\phi-\phi_0)\delta(r-r_0). \tag{F.6}$$

The volume element in spherical coordinates is $dV = dAdr = r^2 dr \sin\theta d\theta d\phi = -r^2 dr d(\cos\theta) d\phi$. dA is the element of area normal to $\vec{r}$. The normalization property is

$$\begin{aligned}\int dV \delta(\vec{r}-\vec{r}_0) &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\;\sin\theta \int_0^{r_m} r^2 \delta(\vec{r}-\vec{r}_0) dr &= 1 \quad (r_m > r_0) \\ & & (\text{F.7}) \\ & &= 0 \quad (r_m < r_0).\end{aligned}$$

r_m is the (arbitrary) radius of a spherical volume centered at the origin.

A very important property applies to the integral of the product of the δ-function with an arbitrary function, say f. For example, if $f = f(x,y)$, then

$$\int dx \int dy f(x,y)\;\delta(x-x_0)\delta(y-y_0) = f(x_0,y_0). \tag{F.8}$$

It must be kept in mind that the volume of integration must include the 'source point' (x_0, y_o) of the δ-function for eqn. F.8 to apply(otherwise the result is zero).

The one-dimensional δ-function has the units of $(\text{length})^{-1}$, while $\delta(\vec{r}-\vec{r}_0)$ has the units of $(\text{length})^{-3}$. Other mathematical forms of the δ-function in terms of the solid angle are given in the next section.

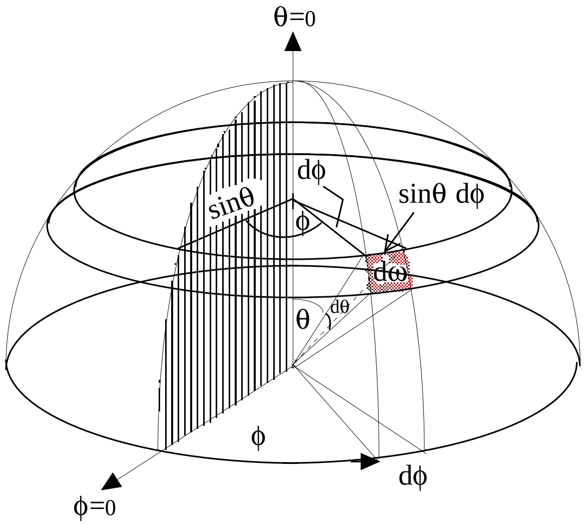

Fig. AF.1. Definition of the solid angle element $d\omega = \sin\theta d\theta d\phi$.

AF.3 The Solid Angle

The *solid angle* ω is defined as the ratio of the area A cut out of a spherical surface (see Fig. AF.1) to the square of the radius of the sphere, i. e. $\omega = A/r^2$. The units of ω are *steradians* $[sr]$. There are 2π *sr* in a hemisphere, and 4π *sr* in a full sphere. We are usually interested in a small (differential) element of solid angle, $d\omega$. As shown in Fig. AF.1, $d\omega$ is expressed in spherical-polar coordinates as $d\omega = dA/r^2$. Since $dA = r^2 \sin\theta d\theta d\phi$

$$d\omega = \sin\theta \; d\theta \; d\phi. \tag{F.9}$$

The integral of eqn. F.9 over the sphere, that is over 4π steradians, is

$$\int_{4\pi} d\omega \equiv \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta = 4\pi. \tag{F.10}$$

Often we consider a solar *beam* (§2.2) of light travelling in a particular direction. This direction is called the *propagation direction* and is specified by a unit vector $\hat{\Omega}_0$, which points in the direction (θ_0, ϕ_0). If we consider a general direction, described by the unit vector $\hat{\Omega}(\theta, \phi)$, a beam is a radiative energy flow which is zero for all directions except $\hat{\Omega}_0$. Thus, we can use a *two-dimensional* δ-function $\delta(\hat{\Omega} - \hat{\Omega}_0)$ to specify this energy flow. In spherical-polar coordinates we have

$$\delta(\hat{\Omega} - \hat{\Omega}_0) = \delta(\cos\theta - \cos\theta_0)\delta(\phi - \phi_0) \tag{F.11}$$

where $\hat{\Omega}_0$ is specified by the angles (θ_0, ϕ_0). The normalization property of the δ-function in eqn. F.11 is

$$\int_{4\pi} d\omega\delta(\hat{\Omega} - \hat{\Omega}_0) = 1. \tag{F.12}$$

While $\delta(\hat{\Omega} - \hat{\Omega}_0)$ is non-dimensional, it will be convenient to think of it as having the 'unit' of inverse steradians $[sr^{-1}]$.