Appendix H

Electromagnetic Radiation: The Plane Wave

We review in this appendix some basic aspects of light. We use *light* as a shorthand for *electromagnetic radiation*, and do not mean to imply *visible light*, which occupies only a small portion of the electromagnetic spectrum. Some simple mathematical fundamentals are provided in Appendix F, including a discussion of elementary concepts such as coordinate systems, the Dirac delta-function, and the solid angle. In this section we restrict our attention to a review of the plane wave, and its polarization properties. More advanced topics concerning the Stokes vector representation, partial polarization and the Mueller matrix are described in Appendix I.

AH.1 Plane Electromagnetic Waves

Light is an electromagnetic phenomenon, along with gamma-rays, x-rays, and radio waves. It is described by solutions of the famous set of equations of J. C. Maxwell, formulated in 1865. These equations in differential form and in *mksa* units for an isotropic, homogeneous, source-free medium, are

$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} \text{ (a)}; \qquad \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \text{ (b)};$$

$$\nabla \cdot \vec{H} = 0 \text{ (c)}; \qquad \nabla \cdot \vec{E} = 0 \text{ (d)}. \tag{H.1}$$

$\nabla\times$ and $\nabla\cdot$ denote the curl and divergence operators, respectively.†

† The relationships between the electric and magnetic field quantities, and the medium properties are called the *constitutive relations*, and are included in the equation set, H.1. See Stratton, J.

$\vec{E}$ and $\vec{H}$ are the *electric* and *magnetic fields*, and t is time. ϵ is the *permittivity*, σ is the *conductivity*, and μ is the *magnetic permeability*, all properties of the medium. A net charge of zero throughout the medium is assumed. The basis of these equations and the medium properties are described in various texts.

A solution of these coupled partial differential equations is sought for this source-free case in which both $\vec{E}$ and $\vec{H}$ are functions of a single spatial variable, and time. Let us assume a purely *dielectric medium*, for which the conductivity σ is zero. Taking the curl† of eqns. H.1a and H.1b and using the vector identity $\nabla \times (\nabla \times \vec{a}) = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$ together with eqns. H.1c and H.1d, we find that both $\vec{E}$ and $\vec{H}$ satisfy the same second-order wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0; \quad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{H} = 0 \quad \text{where } c = \frac{1}{\sqrt{\mu\epsilon}}. \tag{H.2}$$

c is the speed of propagation in the medium. In a vacuum, the speed of light is $c_o = 1/\sqrt{\mu_o \epsilon_o} = 2.9979 \times 10^8$ $[m \cdot s^{-1}]$. The subscript o denotes the vacuum value. It can readily be shown that *plane waves* of the form

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \Re\left\{\vec{E}_0 e^{i(k\hat{\Omega}\cdot\vec{r} - \omega t)}\right\} \\ \vec{H}(\vec{r}, t) &= \Re\left\{\vec{H}_0 e^{i(k\hat{\Omega}\cdot\vec{r} - \omega t)}\right\} \end{aligned} \tag{H.3}$$

are solutions of eqn. H.2. Here $i = \sqrt{-1}$ is the imaginary unit, $\Re$ denotes the real part, and $\vec{E}_0$ and $\vec{H}_0$ are complex constant vectors. The unit vector $\hat{\Omega}$ points in the propagation direction of the plane wave. $k = \omega/c$ is the *wavenumber* [cm^{-1}], and ω is the *angular frequency* [rad·s^{-1}], related to the ordinary frequency ν, [cycles·s^{-1}] or $[Hz]$, by $\omega = 2\pi\nu$. These solutions are called plane waves because at any fixed time t they have the same value at each point in any plane normal to $\hat{\Omega}$, i. e. at any fixed time t, $\vec{E}(\vec{r}, t)$ and $\vec{H}(\vec{r}, t)$ are constant vectors in each plane defined by $\hat{\Omega} \cdot \vec{r} =$ constant.

Note that we have restricted our attention to harmonic plane waves having a sinusoidal variation in time and space. According to eqn. H.3, each Cartesian component of $\vec{E}$ and $\vec{H}$ will be of the general form (with

A., *Electromagnetic Theory*, McGraw-Hill Book CO., New York, 1941. Jackson, J. D. *Classical Electrodynamics*, New York, Wiley, 1975. A good modern text is Griffiths, D. J., *Introduction to Electrodynamics*, Prentice-Hall, 1981.

† For readers unfamiliar with vector analysis, see for example, Edwards, J. and D. E. Penney, Calculus and Analytic Geometry, 3rd ed., Prentice-Hall, Englewood CLiffs, N.J., Chapter 17.

j denoting either x, y, or z)

$$\begin{aligned} E_j(\vec{r},t) &= e_j \cos(k\hat{\Omega}\cdot\vec{r} - \omega t + \delta_j) \\ H_j(\vec{r},t) &= h_j \cos(k\hat{\Omega}\cdot\vec{r} - \omega t + \phi_j) \end{aligned} \tag{H.4}$$

where e_j and h_j are arbitrary real coefficients, and δ_j and ϕ_j are arbitrary phase angles.

The harmonic plane waves in eqns. H.3 are solutions of the wave equation H.2 for arbitrary values of $\vec{E}_0$ and $\vec{H}_0$. But these solutions must also satisfy Maxwell's equations. Substituting eqns. H.3 in eqns. H.1a and H.1b (with $\sigma = 0$), we find that

$$\sqrt{\mu}\,\hat{\Omega}\times\vec{H}_0 = \sqrt{\epsilon}\,\vec{E}_0; \qquad \sqrt{\epsilon}\,\hat{\Omega}\times\vec{E}_0 = \sqrt{\mu}\vec{H}_0 \tag{H.5}$$

from which it follows that $\vec{E}_0\cdot\vec{H}_0 = 0$, and that both $\vec{E}_0$ and $\vec{H}_0$ are orthogonal to the propagation direction $\hat{\Omega}$. In other words, $\vec{E}_0$, $\vec{H}_0$, and $\hat{\Omega}$ form a right-handed triad.

If we now choose the coordinate system such that $\hat{\Omega}$ is along the positive z-axis, we can write

$$\vec{E} = \vec{E}_{||} + \vec{E}_{\perp}; \qquad \vec{E}_{||} = E_{||}\hat{e}_{||}; \qquad \vec{E}_{\perp} = E_{\perp}\hat{e}_{\perp} \tag{H.6}$$

$$\vec{H} = \vec{H}_{||} + \vec{H}_{\perp}; \qquad \vec{H}_{||} = \sqrt{\frac{\epsilon}{\mu}}\hat{e}_z\times\vec{E}_{||}; \qquad \vec{H}_{\perp} = \sqrt{\frac{\epsilon}{\mu}}\hat{e}_z\times\vec{E}_{\perp}. \tag{H.7}$$

Here each of the components $E_{||}$, $E_{\perp}$, $H_{||}$, and $H_{\perp}$ satisfies the wave equation, and $\hat{e}_{\perp}$, $\hat{e}_{||}$, and $\hat{e}_z$ are unit vectors forming a right-handed triad

$$\hat{e}_{\perp}\cdot\hat{e}_{||} = \hat{e}_{\perp}\cdot\hat{e}_z = \hat{e}_{||}\cdot\hat{e}_z = 0, \qquad \hat{e}_{\perp}\times\hat{e}_{||} = \hat{e}_z. \tag{H.8}$$

$E_{||}$ and $E_{\perp}$ are the electric field components parallel and perpendicular to a plane which contains the z-axis, and whose orientation is otherwise arbitrary.†

From eqns. H.3 and H.6–H.7, it follows that

$$E_{||} = \Re\{\mathcal{E}_{||}\}; \qquad E_{\perp} = \Re\{\mathcal{E}_{\perp}\}. \tag{H.9}$$

where the complex amplitudes $\mathcal{E}_{||}$ and $\mathcal{E}_{\perp}$ are given by

$$\mathcal{E}_{||} = a_{||}\exp[i(kz - \omega t + \delta_{||})] \tag{H.10}$$

$$\mathcal{E}_{\perp} = a_{\perp}\exp[i(kz - \omega t + \delta_{\perp})]. \tag{H.11}$$

† This plane will become the plane of incidence when we consider interactions with interfaces, and the scattering plane when we consider interactions with scattering particles.

Here $a_{||}$ and $a_\perp$ are the electric field *amplitudes* and $\delta_{||}$ and $\delta_\perp$ are the *phase angles.* Similar forms can be derived for the magnetic components.

We define the wave number in a vacuum, $k_o \equiv \omega/c_o \equiv 2\pi/\lambda_o$, where λ_o is the *vacuum wavelength.* Then we can express eqns. H.9–H.11 in a more convenient form

$$E_{||,\perp} = \Re\left\{a_{||,\perp}\exp\left\{i\left[(k_o mz - \omega t) + \delta_{||,\perp}\right]\right\}\right\} \tag{H.12}$$

where $m \equiv c_o/c = \lambda_o/\lambda = k/k_o = \sqrt{\epsilon_o\mu_o/\epsilon\mu}$ is the *index of refraction* of the medium, the ratio of the propagation speed *in vacuo* to that in the medium.† These solutions apply to an ideal *harmonic, monochromatic* (single frequency) plane wave of infinite spatial extent ($-\infty < x, y, z < +\infty$) traveling in the positive z-direction. m is often written as a complex quantity, $m \equiv m_r + im_i$. The value of m_r varies slightly with frequency in natural media: in air it is very close to unity – for example, m_r ($\lambda = 1\ \mu m$) $= 1.0 + 2.892 \times 10^{-4}$. In pure water, m_r ($\lambda = 486\ nm$) $= 1.3371$.

The solution for a conducting medium ($\sigma \neq 0$) is worked out in Problem H.1. In this case, the wave is *damped* or *attenuated* along the propagation direction. The solution can be expressed mathematically in the same form as eqns. H.12. In this case the appearance of a 'damping factor' $\exp(-k_o m_i z)$ multiplying eqn. H.12 shows that the presence of a finite conductivity is associated with *absorption* along the wave direction.

AH.2 Energy Transfer

Light waves transmit energy. It is this feature that makes it possible to detect light away from sources, and it explains how the sun warms the earth and ultimately sustains life. The rate at which energy is transported by light is expressed by the *Poynting vector* $\vec{S}$. This quantity is related to the electric and magnetic field vectors, $\vec{E}$ and $\vec{H}$ through $\vec{S} = \vec{E} \times \vec{H}$. This expression gives both the magnitude and direction of instantaneous energy flow. In other words, $\vec{E} \times \vec{H}$ is the *radiative power* per unit area carried along the wave direction.

For time-harmonic plane-wave solutions it follows from eqns. H.6–H.9

† There are actually *two* light speeds to consider: the *phase speed*, $v_p = c = \omega/k$, and the *group speed*, $v_g = \partial\omega/\partial k$. Since $k = n(\omega)\omega/c_0$, and $m(\omega)$ is generally a function of frequency, ω (that is to say, the medium is *dispersive*) then $v_p \neq v_g$. However in a *non-dispersive medium*, $v_p = v_g$.

that

$$\vec{S} = \vec{E} \times \vec{H} = \sqrt{\frac{\epsilon}{\mu}}\,[E_{||}\hat{e}_{||} + E_{\perp}\hat{e}_{\perp}] \times [E_{||}\hat{e}_z \times \hat{e}_{||} + E_{\perp}\hat{e}_z \times \hat{e}_{\perp}]$$

$$= \sqrt{\frac{\epsilon}{\mu}}\,[E_{||}E_{||} + E_{\perp}E_{\perp}]\hat{e}_z = \sqrt{\frac{\epsilon}{\mu}}\,[\Re(\mathcal{E}_{||})\Re(\mathcal{E}_{||}) + \Re(\mathcal{E}_{\perp})\Re(\mathcal{E}_{\perp})]\hat{\Omega}_0. \quad \text{(H.13)}$$

Here $\hat{\Omega}_0$ is the propagation vector of the wave. We are seldom interested in the instantaneous value of $\vec{S}$. Of greater interest is the *time-averaged value*

$$\langle\vec{S}\rangle = \frac{1}{\langle t\rangle}\int_0^{\langle t\rangle} dt\vec{S}(t) \quad \text{(H.14)}$$

where $\langle t\rangle$ is the averaging time. For a periodic function, $\langle t\rangle$ is an integral number of *wave periods*, where one period is $1/\nu$. It is shown in Problem H.2 that the time average of the product of two time-harmonic functions of the same periodicity is

$$\langle\Re\{a(t)\}\cdot\Re\{b(t)\}\rangle = \frac{1}{2}\Re\{ab^*\} = \frac{1}{2}\Re\{a^*b\} \quad \text{(H.15)}$$

where $a(t)$ and $b(t)$ both are of the form in eqns. H.10 and H.11. The asterisk denotes complex conjugation. Using this result in eqn. H.13, we find that the flow in the general direction $\hat{\Omega}$ is

$$\langle\vec{S}\rangle = \frac{m}{2\mu c_o}\left\{\frac{\epsilon}{2}[\mathcal{E}_{||}\cdot\mathcal{E}_{||}^* + \mathcal{E}_{\perp}\cdot\mathcal{E}_{\perp}^*]\right\}\delta(\hat{\Omega}-\hat{\Omega}_0) \quad \text{(H.16)}$$

where we have used $c = 1/\sqrt{\mu\epsilon}$ and $m = c_o/c = \sqrt{\mu\epsilon/\mu_o\epsilon_o}$. The quantity in the curly brackets is the energy density $\mathcal{U} = \mathcal{U}_e + \mathcal{U}_m$ of the plane electromagnetic wave, consisting of the sum of electric field ($\mathcal{U}_e$) and magnetic field ($\mathcal{U}_m$) energy densities. Eqn. H.16 shows that the energy density of the plane electromagnetic wave propagates with velocity $c = c_o/m$ in the z-direction.

Also, eqn. H.16 shows that a plane electromagnetic wave may be considered to have two components

$$I_{||} = (m/2\mu c_o)|\mathcal{E}_{||}|^2 \quad \text{and} \quad I_{\perp} = (m/2\mu c_o)|\mathcal{E}_{\perp}|^2. \quad \text{(H.17)}$$

$I_{||}$ and $I_{\perp}$ are called the *intensity components*.† Eqn. H.16 tells us that

† Here we are using the physicists definition of intensity. In fact this is closer to our definition of the flux, or irradiance (Chapter 2).

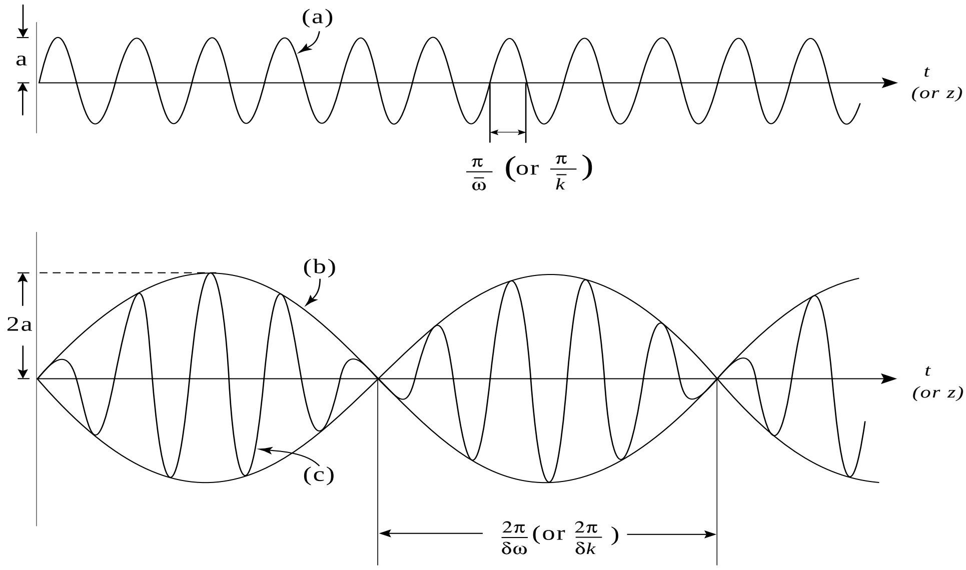

Fig. AH.1. A simple wave packet. (a) The wave $a\cos(\bar{k}z - \bar{\omega}t)$. (b) The wave packet $2a\cos\left[\frac{1}{2}(z\delta k - t\delta\omega)\right]\cos(\bar{k}z - \bar{\omega}t)$. The ordinate represents one of the two independent variables (t or z) while the other is kept constant.

the average radiative power is $(I_{\|} + I_{\perp})\ \delta(\hat{\Omega} - \hat{\Omega}_0)$. The fact that light waves have two independent components accounts for the phenomenon of *polarization*. It distinguishes light waves from *scalar waves*, such as sound waves in liquids or gases, which have only a single energy-carrying component.

AH.3 Addition of Plane Waves

The monochromatic plane wave solutions are *elementary* solutions to Maxwell's equations. Clearly, they are idealizations. Any real wave is a linear superposition of monochromatic plane waves of different frequencies, directions, and phases. If all waves in a group have almost the same frequency, we have a *wave packet*. Consider a wave packet consisting of only two waves, both propagating along the z-axis, and having slightly different frequencies and wave numbers. Let the waves have the same amplitude $a_{\|}$, and consider only one polarization component, say $E_{\|}$. The total electric field is the coherent sum of the individual waves, i. e.

$$E_{||}(z,t) = \Re\left\{a_{||}e^{i(kz-\omega t+\delta_1)} + a_{||}e^{i[(k+\delta k)z-(\omega+\delta\omega)t+\delta_2]}\right\} \tag{H.18}$$

where $\delta\omega$ and δk are the (small) differences in frequencies and wave numbers, and δ_1 and δ_2 are the respective phase angles. We may combine the two terms by using the well-known relationship between the cosine-function and the complex exponentials. The result is

$$E_{||}(z,t) = 2a_{||}\cos[(1/2)(z\delta k - t\delta\omega + \delta')]\Re\left\{e^{i(\overline{k}z-\overline{\omega}t+\overline{\delta})}\right\} \tag{H.19}$$

where $\overline{\omega} = \omega + \frac{1}{2}\delta\omega$, $\overline{k} = k + \frac{1}{2}\delta k$, and $\overline{\delta} = (\delta_1 + \delta_2)/2$. These are the mean angular frequency, the mean wave number, and the mean phase angle, respectively. δ' is the phase angle difference $\delta_1 - \delta_2$. The resultant wave is a plane wave of angular frequency $\overline{\omega}$ and wavelength $2\pi/\overline{k}$ propagating in the z-direction. However, the amplitude of the wave is not constant, but varies with time and position, between the values of $2a_{||}$ and zero (see Fig. AH.1).

This is a mathematical description of the phenomenon of *beats*. The two waves change from being totally in phase (where *constructive interference* occurs) to being totally out of phase (where *destructive interference* occurs). If we set the two frequencies or wave numbers equal, we have two monochromatic plane waves with differing phases, i. e.

$$E_{||}(z,t) = 2a_{||}\cos\left[(1/2)(\delta_1 - \delta_2)\right]\Re\left\{e^{i(kz-\omega t+\overline{\delta})}\right\}. \tag{H.20}$$

When the phases are equal, $\delta_1 = \delta_2$, the amplitude in eqn. H.20 has its maximum value, $2a_{||}$. Again we have constructive interference for in-phase waves. For $\delta_1 - \delta_2 = \pm n\pi$ $(n = 1, 2, \cdots)$, we obtain a zero amplitude for out-of-phase waves, and the destructive interference is complete.

AH.4 Standing Waves

We now consider the superposition of two plane waves travelling in *opposite* directions. This will lead us to the concept of a *standing wave*, a topic of importance to the subject of blackbody radiation. We imagine two oppositely-directed waves of the same frequency, phase and amplitude (the latter we set equal to unity). Again, consider only one component (say the parallel component) of the electric field. The total E-field component is

$$E_{||}(z,t) = \Re\Big\{e^{i(kz-\omega t-\pi/2)} + e^{i(-kz-\omega t-\pi/2)}\Big\} \tag{H.21}$$

where we have chosen the phase $\delta = -\pi/2$ for convenience. Using the relationship $\cos kX = (1/2)[\exp(ikX) + \exp(-ikX)]$, we write eqn. H.21 as

$$E_{||}(z,t) = 2\cos(kz + \pi/2)\Re\Big\{e^{-i\omega t}\Big\} = 2\sin(kz)\cos(\omega t).$$

The result is a wave that neither moves forward or backward. It vanishes at values of z for which $\sin(kz) = 0$, that is, where $kz = n\pi$ $(n = 0, 1, \cdots)$. In between these *nodes*, the disturbance vibrates harmonically with time. The maxima are located at the *anti-nodes*, at $kz = n\pi/2$ $(n = 1, 3, \cdots)$.

For a standing wave located in a finite cavity, the electric field must vanish at the boundaries, say at $z = 0$ and at $z = L$. The nodes will of course correspond with the boundaries, so that $k = n\pi/L$ $(n = 0, 1, \cdots)$. For example, the two lowest-order *wave-modes* are given by

$$E^{(1)}(z,t) = 2\cos(\pi z/L)\cos(\omega t); \qquad E^{(2)}(z,t) = 2\cos(2\pi z/L)\cos(\omega t).$$

The $n = 1$ wave-mode is fixed at the two ends; the $n = 2$ wave-mode is fixed at both ends and in addition is fixed at the center, $z = L/2$. Higher-order wave-modes $E^{(n)}$ have $n+1$ nodes, etc.

In a three-dimensional cavity (taken to be cubic of sides L for convenience), there are three independent components (actually six, taking into account the perpendicular component). Each has its own wave number, so that

$$k_x = n_x\pi/L; \quad k_y = n_y\pi/L; \quad k_z = n_z\pi/L \qquad (n_x, n_y, n_z = 0, 1, \cdots).$$

In vector notation, we write $\vec{k} = \pi\vec{n}/L$ where $\vec{n}$ is a vector in a three-dimensional pseudo-space with Cartesian components n_x, n_y and n_z.

The above results are applicable to the study of blackbody radiation and is used in the derivation of the *Planck distribution* in §4.3. A radiation field may be thought of as a system of standing waves in a large cavity, or *hohlraum*. The cavity 'walls' are unimportant except for establishing the boundary conditions. In the quantum theory each standing wave may be associated with a *photon*, a particle of light having a quantized energy and momentum given by

$$\begin{aligned}\text{photon energy} &= \mathcal{E}_p = h\nu = \frac{hc}{\lambda} = \frac{h}{2\pi}\omega = \frac{h}{2\pi}c|\vec{k}| \\ \text{photon momentum} &= \mathcal{P}_p = \frac{h\nu}{c} = \frac{h}{\lambda} = \frac{h}{2\pi}|\vec{k}|.\end{aligned}$$

where h is Planck's constant=6.63×10^{-34} $[J \cdot s]$.

In this appendix we found that the linear superposition of electromagnetic fields leads to the phenomena of beating, interference and standing waves. These are all results of *coherent addition* of light waves, and is to be contrasted with the very different situation of *incoherent addition*. It is the latter situation we are mainly concerned with in this book.

AH.5 Polarization

We now consider the way in which the electric field vector of a plane wave varies in space and time. Defining the variable part of the phase factor of eqns. H.9–H.11 as $\phi = kz - \omega t$, we may write the electric field components as

$$E_{||} = a_{||}\cos(\phi + \delta_{||}); \qquad E_{\perp} = a_{\perp}\cos(\phi + \delta_{\perp}). \tag{H.22}$$

We can determine how $\vec{E}$ varies in space by eliminating ϕ. It is easily shown that

$$\left(\frac{E_{||}}{a_{||}}\right)^2 + \left(\frac{E_{\perp}}{a_{\perp}}\right)^2 - 2\frac{E_{||}E_{\perp}}{a_{||}a_{\perp}}\cos\delta = \sin^2\delta \tag{H.23}$$

where $\delta \equiv \delta_{||} - \delta_{\perp}$. This is the equation of an ellipse, which is inscribed into a rectangle whose sides are parallel to the coordinate axes, and whose lengths are $2a_{||}$ and $2a_{\perp}$ (see Fig. AH.2).

At a given point in space, the tip of the electric field vector will therefore trace out an ellipse – the wave is said to be *elliptically polarized.* The properties of the ellipse are determined by three quantities: either $a_{||}, a_{\perp}$ and $\delta = \delta_{||} - \delta_{\perp}$; or by the major and minor axes, a and b, and the angle ψ. The latter is the angle the major axis makes with the

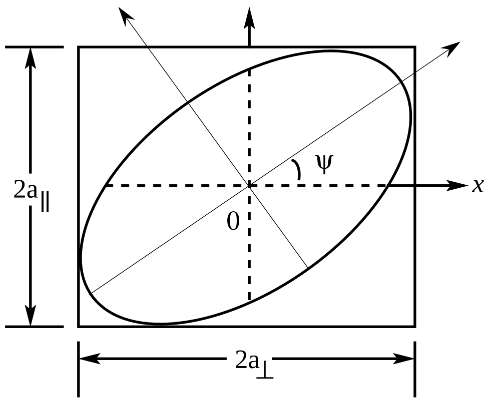

Fig. AH.2. Elliptically polarized wave. The vibrational ellipse for the electric vector. The ellipse is inscribed into a rectangle whose sides are parallel to the coordinate axes whose lengths are $2a_{||}$ and $2a_{\perp}$. The ellipse touches the sides at the points $(\pm a_{||}, \pm a_{\perp}\cos\delta)$ and $(\pm a_{||}\cos\delta, \pm a_{\perp})$.

horizontal (parallel axis) as shown in Fig. AH.2. It may be shown that these quantities are related to the first set by

$$a^2 + b^2 = a_{||}^2 + a_{\perp}^2; \qquad \pm ab = a_{||}a_{\perp}\sin\delta;$$

$$\tan 2\psi = (\tan 2\alpha)\cos\delta; \qquad \tan\alpha = \frac{a_{||}}{a_{\perp}}. \tag{H.24}$$

AH.6 Polarization: linear and circular

The special cases of linear and circular polarization occur when the ellipse in eqn. H.23 degenerates into either a straight line or a circle. When the phase difference of the two components is an integral multiple of π, that is when $\delta = \delta_{||} - \delta_{\perp} = m\pi$ for $(m = 0, \pm 1, \pm 2, \cdots)$, eqn. H.23 yields

$$\frac{E_{\perp}}{E_{||}} = (-1)^m \frac{a_{\perp}}{a_{||}}.$$

In this case, $\vec{E}$ is *linearly polarized.* The two components bear a constant ratio to one another. Considering the time-dependent factor ϕ

(see eqn. H.22), we see that the $\vec{E}$-vector oscillates in magnitude (with angular frequency ω) along a straight line, from the value $-a_{||}$ to $+a_{||}$. When the components have equal magnitude, $a_{||} = a_{\perp} = a$, and in addition the phase angles are in quadrature, that is $\delta = \delta_{||} - \delta_{\perp} = m\pi/2$ where $m = (\pm 1, \pm 3, \pm 5, \cdots)$, eqn. H.23 reduces to the equation for a circle, i. e.

$$E_{||}^2 + E_{\perp}^2 = a^2.$$

Additional information on the *Stokes-vector* representation of light, and other advanced topics, is given in Appendix I and in other texts.† In the natural environment light is *partially-polarized* or in some limiting situations, *unpolarized*. Simply stated, the latter means that there is no preference between the parallel- and perpendicular-directions, and no permanent phase relationships exist between these two components. Sunlight, diffuse visible light emanating from an optically-thick cloud cover, and thermal IR emission are important examples of (nearly) unpolarized light. Rayleigh scattering from a clear sky is a counter-example, as the degree of linear polarization of scattered light can be quite high. Despite its importance in some applications, we will ignore polarization on the grounds that we are mainly concerned with the energy flow, rather than the accurate intensity distribution. This is called the *scalar approximation*. Even though caution is advised, it often provides reasonably accurate results even for the directional distribution of radiation. In addition there are ways to estimate the polarization by making first-order corrections to scalar solutions.

AH.7 Problems

H.1. Consider a plane electromagnetic wave propagating in the z-direction through an isotropic, homogeneous medium with conductivity σ and permittivity ϵ. For this geometry Maxwell's equations simplify to

† Plane waves, polarization, and the Stokes parameters are discussed in the following references: Born, M. and E. Wolf, *Principles of Optics*, Chapter 1, MacMillan, New York, 1964. Coulson, K. L.,*Polarization and Intensity of Light in the Atmosphere*, A. Deepak Publ., Hampton, Va., 1988: Kliger, D. S., J. W. Lewis, and C. E. Randall, *Polarized Light in Optics and Spectroscopy*, Academic Press, Boston, 1990. A practical non-mathematical approach is found in Shurcliff, W. A. and S. S. Ballard, *Polarized Light*, Van Nostrand, Princeton, 1964; An influential journal review is Hansen, J. E. and L. D. Travis, Light Scattering in Planetary Atmospheres, *Space Sci. Rev.*, **16**, 527-610, 1974.

$$\frac{\partial^2 E_{||}}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_{||}}{\partial t^2} + \mu\sigma \frac{\partial E_{||}}{\partial t}$$

$$\frac{\partial^2 E_{\perp}}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_{\perp}}{\partial t^2} + \mu\sigma \frac{\partial E_{\perp}}{\partial t}.$$

(a) Show that the electric field strength diminishes along the beam according to the exponential *Extinction Law* (§2.7), that is, the above set of equations has a solution of the form

$$E_{||} = a_{||} \exp\{i[(\omega t - k_o m_r z)] - k_o m_i z\}$$

where

$$k_o = \frac{2\pi}{\lambda_o} = \frac{\omega}{c_o} = \omega\sqrt{\mu_o \epsilon_o}$$

and m_r and m_i are the real and imaginary parts of the complex index of refraction.

(b) Find the expressions for the two quantities, m_r and m_i, and for the speed of light in the medium in terms of the electric and magnetic properties of the medium. Show that the absorption coefficient $\alpha = k_o n_i$ is given by

$$\alpha = \omega \sqrt{\frac{\epsilon\mu}{2}\left[-1 + \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}\right]}.$$

H.2. (a) Show that the real part of the time-average of the product of two complex quantities, A^* and B^* (having the same periodicity) is given by

$$< \Re(A) \cdot \Re(B) >= \frac{1}{2}\Re(AB^*). \tag{H.25}$$

(b) Solve for the H-components of the plane wave travelling in a dielectric medium. From these expressions, show that the Poynting vector is given by eqn. H.16.