Appendix I

Representations of Polarized Light

AI.1 Stokes Parameters

In addition to the frequency, three independent quantities are needed to completely specify a time-harmonic electromagnetic plane wave. Since the quantities used in Appendix H are combinations of amplitudes and angles, which have different units, it is more convenient to use quantities having the same dimensions. In 1852 G. G. Stokes introduced his four parameters

$$I = a_{||}^2 + a_{\perp}^2; \qquad Q = a_{||}^2 - a_{\perp}^2; \qquad U = 2a_{||}a_{\perp}\cos\delta; \qquad V = 2a_{||}a_{\perp}\sin\delta. \tag{I.1}$$

Only three of these are independent, since $I^2 = Q^2 + U^2 + V^2$. We already found that I is the energy carried by the wave. The other parameters are related to the angle ψ $(0 \le \psi < \pi)$ specifying the orientation of the ellipse (Fig. AI.2), and the *ellipticity* angle, $\chi(-\pi/4 \le \chi \le \pi/4)$, which is given by $\tan\chi = \pm b/a$. The relationships are as follows:†

$$Q = I\cos 2\chi\cos 2\psi; \qquad U = I\cos 2\chi\sin 2\psi; \qquad V = I\sin 2\chi. \tag{I.2}$$

AI.2 The Poincaré Sphere

Eqns. I.2 provide a simple geometrical representation of all the different states of polarization: Q, U and V may be regarded as the Cartesian coordinates of a point P on a sphere of radius I, such that 2χ and 2ψ are the spherical coordinates of this point (Fig. AI.2). Every possible state of polarization of a plane wave is represented by a point on this *Poincaré Sphere*, developed by H. Poincaré in 1892. A point in the

† See Born and Wolf, Principles of Optics, pp. 24-31.

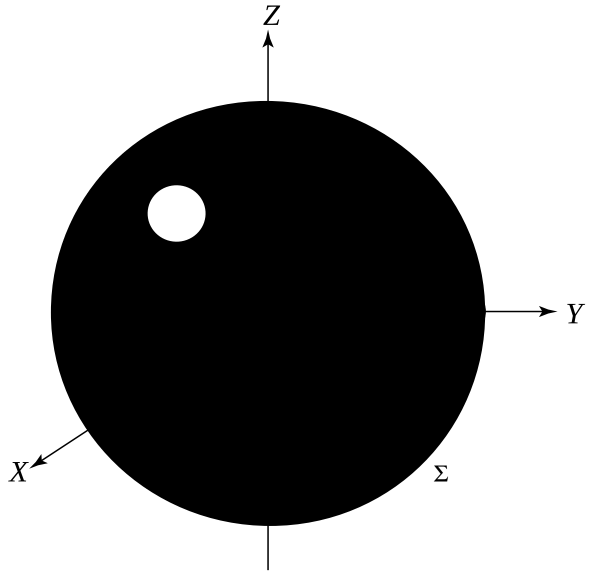

Fig. AI.1. Poincaré's representation of the state of polarization of a monochromatic wave. (The Poincaré sphere).

upper hemisphere (χ positive) represents *right-handed polarization*, that is, when the observer views the wave 'head-on', $\vec{E}$ rotates in a *clockwise* direction. *Left-handed* polarization corresponds to a point in the lower hemisphere (χ negative); when the observer views $\vec{E}$ 'head on', it rotates in a *counter-clockwise* direction. Linear polarization occurs when the phase difference δ is zero, or an integral multiple of π. From eqn. I.1, V is zero, and from eqn. I.2, the z-component of the point P on the Poincaré Sphere is zero. Linear polarization is represented by points in the equatorial plane. For circular polarization, $a_{||} = a_{\perp}$, and $\delta = \pi/2$ or $\delta = -\pi/2$, according to whether the polarization is right- or left-handed. Thus right-handed circular polarization corresponds to the north pole ($Q = U = 0, V = I$), and left-handed circular polarization corresponds to the south pole ($Q = U = 0, V = -I$). *Elliptical polarization* corresponds to a general point on the sphere, other than those in the equatorial plane or at the poles.

The Poincaré Sphere is useful in giving a simple geometrical visualization of the Stokes parameters. It applies only to a light wave which is *perfectly polarized*, an idealization which seldom occurs in nature. We now consider the general situation in which correlation between the two electric field components is not perfect.

AI.3 Partial Polarization and the Incoherency of Natural Light

So far we have assumed that light is a plane wave with constant amplitude and phase difference between the two components. However, a more realistic view is that light is a mixture of plane waves, whose $\vec{E}$-field oscillates over a staggering number of cycles in one second. For example for visible light of $\lambda = 500\ nm$, a wave oscillates at 6×10^{14} cycles in one second. Even a detector with a very short integration time (say, 10^{-4}s) will time-average over many oscillations. The effective Stokes parameters measured by a detector is therefore not the instantaneous values (given by eqn. I.2), but the time-averaged values

$$I = \left\langle a_{||}^2 \right\rangle + \left\langle a_{\perp}^2 \right\rangle \; ; \; Q = \left\langle a_{||}^2 \right\rangle - \left\langle a_{\perp}^2 \right\rangle \; ; \; U = \left\langle 2a_{||}a_{\perp} \cos\delta \right\rangle \; ; \; V = \left\langle 2a_{||}a_{\perp} \sin\delta \right\rangle . \tag{I.3}$$

Generally, a light wave consists of a mixture of waves from different sources, which are statistically uncorrelated over the averaging time of a detector. Suppose we pass the light from such a 'natural' source, e. g. a hot filament, through a filter which passes only a narrow band of frequencies. Even though the frequencies of all the waves are practically equal, the phase angles will differ from one wave to the other. We may visualize the E-components at a point in space as being harmonic in time over immeasurably short time intervals (of the order of $10^{-8} - 10^{-9}\ s$), but 'switching' randomly from one phase angle to another over longer time intervals. If this switching occurs in completely random ways, there will be as many positive phase differences as negative phase differences, or in other words, the time averages of the products $2a_{||}a_{\perp} \cos\delta$ and $2a_{||}a_{\perp} \sin\delta$ will be zero. Similarly, we can visualize the amplitudes being harmonic, and of specific amplitudes over short time intervals, but in a mixture of uncorrelated waves, the average intensities of the two polarization components will be the same, that is, $\langle a_{||}^2 \rangle = \langle a_{\perp}^2 \rangle$. *Thus, for an uncorrelated mixture of plane waves, Q, U, and V all vanish.* This is known as *unpolarized* light. Examples of unpolarized light are direct sunlight, diffuse skylight from an overcast sky, and infrared thermal radiation. However, most scattered light in natural media is partially polarized. It is clear that if some correlation exists between amplitudes or phases, Q, U, and V may be finite, but smaller in value than in the case of a mixture of *coherent* waves. Thus, we see that the difference between coherent and *incoherent* light is the *degree of correlation* between the two E-field components. In this case, the relationship $I^2 = Q^2 + U^2 + V^2$ (valid for fully polarized light; see eqn. I.1) becomes an *inequality*, $I^2 \geq Q^2 + U^2 + V^2$. This property gives us a quantitative measure of the *degree of polarization*, defined as

$$P = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}. \tag{I.4}$$

What is the physical significance of the Stokes parameters? We can relate I, Q, U, and V to a set of ideal measurements, involving a linear polarizer (such as a polaroid filter), and a retardation plate (such as a thin calcite crystal). The polaroid removes the $\vec{E}$-field component of light that passes through in a direction perpendicular to its axis of polarization, and transmits the other component with 100% transmission†. The retardation plate will affect the relative phases of the two components, i. e. it will introduce a relative phase shift, δ. Suppose we have a radiation detector which measures the radiative energy which has passed through a polarizer-retarder combination. It may be shown‡ that the intensity of transmitted light is given by

$$I(\psi, \delta) = (1/2)\left[I' + Q' \cos 2\psi + (U' \cos\delta + V' \sin\delta) \sin 2\psi\right] \tag{I.5}$$

where primed quantities represent the Stokes parameters of the incident light, $\delta = \delta_{||} - \delta_{\perp}$ is the retardation of the $\perp$-component, relative to the $\|$-component, and ψ is the angle of the polarizer axis with the horizontal ($\|$) axis. It is clear from eqn. I.5 that we can use a number of measurements of the incoming beam (varying ψ) to solve for the Stokes parameters of the incident light. If we first consider only a linear polarizer in the beam, so that there is no retardation ($\delta = 0$), and make measurements at $\psi = 0°$, $45°$, $90°$, and $135°$, the first three Stokes parameters may be obtained from these four measurements of $I(\psi, \delta)$:

$$\begin{aligned} I' &= I(0°, 0) + I(90°, 0) \quad &\text{(a)} \\ Q' &= I(0°, 0) - I(90°, 0) \quad &\text{(b)} \\ U' &= I(45°, 0) - I(135°, 0) \quad &\text{(c)}. \end{aligned} \tag{I.6}$$

It is clear from eqn. I.6 that the fourth component V' cannot be measured with a linear polarizer alone: a retarder is needed. Suppose

† Ideally, a polaroid filter would have no effect on that component parallel to the polarization axis, but in all real polaroids, some absorption will take place along this axis also.

‡ The equivalent form of Eqn. I.5 in S. Chandrasekhar, Radiative Transfer, Dover, eqn. 163 (p. 129) has been shown by Hansen and Travis (see endnotes) to have an error in sign. This error arises in the inconsistency between Chandrasekhar's definition of phase difference in his eqn. 154 (p. 28) with the definition of phase difference employed for the Stokes parameters.

we use a polarizer/quarter-wave plate combination. For $\delta = \pi/2$, we get

$$V' = I(45^\circ, \pi/2) - I(135^\circ, \pi/2). \tag{I.7}$$

The physical significance of the Stokes parameters can now be stated in terms of *preferences* as follows: (1) Q gives preference to the $\|$-component over the $\perp$-component; (2) U gives preference to the component making an angle of 45° over that making an angle of 135°; and V gives preference to the 45° component over the 135° component when passed through a polarizer-retarder combination. If unpolarized light were subjected to these measurements, the intensities $I(\psi, \delta)$ would be independent of ψ and δ, so that $Q' = U' = V' = 0$.

If we were to add two polarized light beams together, what is the polarization of the mixture? We found earlier that if we add together two coherent plane waves of the same frequency and amplitude, we obtained an intensity that varies between zero and twice the amplitude of an individual wave. This occurred because of mutual interference, which depended upon the phase angle difference between the two waves. However, if we add together two partially-polarized waves with *no* (time-average) correlation between the phases, *the net result is that the Stokes parameters of the mixture is the sum of the individual Stokes parameters.* This is the most important property of the Stokes parameters. In this book we consider such light mixtures, or in other words, we consider *incoherent light fields.*

Despite our emphasis on incoherent light in this book, it is important to remember that coherent processes are also at work in the natural environment; otherwise we would be deprived of a host of beautiful phenomena, such as rainbows, iridescence, haloes, mirages, etc.† This co-existence of coherent and incoherent light is explained by the notion of *partial coherence*, and the spatial scales over which the various phenomena occur. A natural radiation field is coherent over an inner scale, called the *coherence length, d* (usually $d \approx \lambda$). Thus, light transmitted through a dielectric particle will undergo coherent interaction with its mutual parts, provided the circumference of the particle is of the order of λ. On the other hand, if the particle is much larger than λ, the various beams will behave as if they are refracted and transmitted

† For a lucid description of coherent processes in nature, this classic text should be consulted: M. Minneart, The Nature of Light and Colour in the Open Air, Dover Publications, New York, N. Y., 1954. A new edition of this book was published in 1993 by Springer, with color photos by Pekka Parviainen.

independently‡. In this case, the laws of geometrical optics provide a good description of the overall interaction.

AI.4 The Stokes Vector Representation of Polarized Light

The *Stokes vector* $\vec{I}$ is a four-vector having the four Stokes parameters as its components,

$$\vec{I} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}. \tag{I.8}$$

In view of the linearity property of light fields, the Stokes vector of a mixture of two incoherent light fields whose Stokes vectors are $\vec{I}_1$ and $\vec{I}_2$ is simply $\vec{I} = \vec{I}_1 + \vec{I}_2$, or

$$\vec{I} = \begin{pmatrix} I_1 \\ Q_1 \\ U_1 \\ V_1 \end{pmatrix} + \begin{pmatrix} I_2 \\ Q_2 \\ U_2 \\ V_2 \end{pmatrix} = \begin{pmatrix} I_1 + I_2 \\ Q_1 + Q_2 \\ U_1 + U_2 \\ V_1 + V_2 \end{pmatrix}. \tag{I.9}$$

The additivity principle also tells us that an unpolarized radiation field can be represented as the sum of two *linearly-polarized* fields which have equal E-field components and have their polarization directions normal to one another. Thus, two linearly-polarized incoherent light fields of equal intensity $(I/2)$ add together to give an unpolarized field:

$$\vec{I} = (I/2) \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (I/2) \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = I \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{I.10}$$

Note that the first vector in eqn. I.10 has its polarization direction in the $\|$-direction, so that from eqn. I.6, the component in the 90°-direction is zero. The second vector has a zero component in the 0°-direction, so that Q is the negative of that of the first vector.

It is also easy to see that an arbitrarily-polarized light field may be represented by the sum of an unpolarized (u) and a perfectly-polarized

‡ This assumes that the particle is optically homogeneous. If the particle is inhomogeneous, scattering from irregularities causes the internal radiation field to be multiply-scattered, and mutual interference complicates the description. See §3.2 for more discussion.

(p) light field

$$\vec{I} = \vec{I_u} + \vec{I_p} = \begin{pmatrix} I - \sqrt{U^2 + Q^2 + V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{U^2 + Q^2 + V^2} \\ Q \\ U \\ V \end{pmatrix}. \quad \text{(I.11)}$$

In view of the additivity of Stokes parameters it is easy to see why it is possible to represent any arbitrarily-polarized, incoherent radiation field as the linear sum of an unpolarized part I_u and a 100% polarized part, I_p, $I = I_u + I_p$. The degree of polarization is then written $P = I_p/I$, which gives us a more intuitive interpretation of P than provided by eqn. I.4.

For perfectly-polarized light ($\vec{I_u} = 0$), $\vec{I}$ is a vector whose tip lies on the Poincaré sphere. We may visualize partially-polarized light as a vector $\vec{I_p}$, to which is added a 'smeared-out' component of radius $\vec{I_u}$. Over the averaging time period $< t >$, the tip of the vector $\vec{I_u}$ traces out with equal probability all 4π steradians of the Poincaré sphere.

AI.5 The Mueller Matrix

The action of any optical device on an incoherent light beam can be thought of as producing a Stokes vector which is a linear combination of the Stokes components of the light. Formally, we can represent the effect of an optical device in terms of a *Mueller matrix* operation on $\vec{I}$, or in mathematical terms

$$\vec{I} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \mathbf{M}\vec{I'} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \begin{pmatrix} I' \\ Q' \\ U' \\ V' \end{pmatrix}. \quad \text{(I.12)}$$

The input radiation field components are denoted by primes, and the output radiation field components are unprimed. The components M_{ij} may be derived for various types of polarization analyzers, including polaroid filters (or in general *dichroic linear polarizers*) and retarding plates.† We are often concerned with the action of scattering particles on the state of polarization of an incident radiation field. This can also be represented as a linear matrix operator, called the *scattering matrix*, $\mathbf{S}$, whose elements depend upon the angle Θ between the incident and scattered wave (the scattering angle), i. e., $S_{ij} = S_{ij}(\Theta)$. In addition

† see Coulson, Polarized Light, pp. 577–584.

S_{ij} depends upon the light-interaction properties of the particles. For the simplest type of scattering, i. e. *Rayleigh scattering*, the scattering matrix is given by

$$\mathbf{S}_{Ray}(\Theta) = \frac{3\sigma}{4\pi} \begin{pmatrix} 1+\cos^2\Theta & \cos^2\Theta - 1 & 0 & 0 \\ \cos^2\Theta - 1 & 1+\cos^2\Theta & 0 & 0 \\ 0 & 0 & 2\cos\Theta & 0 \\ 0 & 0 & 0 & 2\cos\Theta \end{pmatrix} \tag{I.13}$$

where σ is the scattering coefficient, defined in Chapter 3.

The radiation field in atmospheres and oceans can be highly polarized. For example, for clear skies or pure oceans where Rayleigh scattering dominates the radiative transfer, eqn. I.13 shows that for scattering angles near $\Theta = \pi/2$, there is 100% linear polarization for $\Theta = \pi/2$. However, in reality there are slight deviations from this idealized Rayleigh scattering so that the light is about 96% polarized for $\Theta = \pi/2$. (The presence of aerosols reduces this number to no more than 80% in actual cloud-free situations.) Reflection from water or ice surfaces can also lead to high linear polarizations. However, the elliptic component V is always very small, and it is seldom necessary to specify all four Stokes parameters. In fact, since I conveys the information on the energy carried by the field, it is often permissible to ignore the Q and U components as well. This is the principal approximation of this book. We note, however, that although I 'carries the energy', it is sometimes necessary to solve the full *vector* equation (for the Stokes' parameters) to calculate it properly. The *scalar* equation is in many cases adequate as can be confirmed by comparison between *vector* and *scalar* solutions.

AI.6 Problems

I.1. (a) Find the Stokes vector for Rayleigh-scattered light from a small volume element dV having a concentration of n molecules. Use the equation $\vec{I} = n\, dV\, \mathbf{S}_{Ray}\vec{I'}$, where $\mathbf{S}_{Ray}$ is given by eqn. I.13. Assume that the solar intensity is unpolarized and given by

$$\vec{I'} = \begin{pmatrix} F^s\delta(\cos\theta_0 - \cos\theta)\delta(\phi_0 - \phi) \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{I.14}$$

where F^s is the solar flux $[W \cdot m^{-2}]$ and (θ_0, ϕ_0) is the direction of the incoming solar beam.

(b) Describe the state of polarization for Rayleigh scattered light evaluated at the scattering angles $\Theta = \pi/2$ and $\Theta = 0$.

I.2. Devise a number of 'thought experiments' to find the elements of the Mueller matrix for the following optical instruments:

(a) an ideal linear polarizer, e. g. a polaroid filter, with its axis along the horizontal ($\|$)-axis.

(b) the same as (a) but with its axis along the perpendicular ($\perp$)-axis.