Appendix J

Scaling Transformation for Anisotropic Scattering

We will show that the transfer equation is invariant under certain scale changes of the optical depth and the phase function. The so-called $\delta - N$ method, discussed in §6.8.4, turns out to be one such invariant scaling transformation.

We start with the general radiative transfer equation for the total intensity which in slab geometry may be written

$$u\frac{dI(\tau,\hat{\Omega})}{d\tau} = I(\tau,\hat{\Omega}) - \frac{a}{4\pi}\int_{4\pi} dw' \, p(\hat{\Omega}',\hat{\Omega})I(\tau,\hat{\Omega}') \tag{J.1}$$

where we have ignored the thermal emission term. If we define a kernel

$$G(\hat{\Omega}',\hat{\Omega}) \equiv \frac{1}{4\pi}[-ap(\cos\Theta) + 4\pi\delta(\hat{\Omega}' - \hat{\Omega})] \tag{J.2}$$

then we may rewrite eqn. J.1 as

$$u\frac{dI(\tau,\hat{\Omega})}{d\tau} = \int_{4\pi} G(\hat{\Omega}',\hat{\Omega})I(\tau,\hat{\Omega}')d\omega'. \tag{J.3}$$

Now, by introducing a new optical depth, $\hat{\tau}$, and a new kernel, $\hat{G}$, through

$$\tau = \beta\hat{\tau} \tag{J.4}$$
$$G = \beta^{-1}\hat{G} \tag{J.5}$$

we find that eqn. J.3 becomes

$$u\frac{dI(\hat{\tau},\hat{\Omega})}{d\hat{\tau}} = \int_{4\pi} \hat{G}(\hat{\Omega}',\hat{\Omega})I(\hat{\tau},\hat{\Omega}')d\omega' \tag{J.6}$$

In view of the definition of G (eqn. J.2) we may rewrite eqn. J.6 as

$$u\frac{dI(\hat{\tau},\hat{\Omega})}{d\hat{\tau}} = I(\hat{\tau},\hat{\Omega}) - \frac{\hat{a}}{4\pi}\int_{4\pi}\hat{p}(\hat{\Omega}',\hat{\Omega})I(\tau,\hat{\Omega}')d\omega' \tag{J.7}$$

where

$$\begin{aligned}\hat{G}(\hat{\Omega}',\hat{\Omega}) &= \frac{1}{4\pi}[-\hat{a}\hat{p}(\cos\Theta) + 4\pi\delta(\hat{\Omega}'-\hat{\Omega})] = \beta G(\hat{\Omega}',\hat{\Omega}) \\ &= \frac{1}{4\pi}[-\beta a p(\cos\Theta) + 4\pi\beta\delta(\hat{\Omega}'-\hat{\Omega})]\end{aligned} \tag{J.8}$$

which implies

$$\hat{a}\hat{p}(\cos\Theta) = [\beta a p(\cos\Theta) + 4\pi(1-\beta)\delta(\hat{\Omega}'-\hat{\Omega})]. \tag{J.9}$$

If we now require the scaled phase function to be normalized to unity as usual, then integration of eqn. J.9 over 4π steradians yields

$$\hat{a} = a\beta + (1-\beta) \tag{J.10}$$

or

$$1 - \hat{a} = \beta(1-a). \tag{J.11}$$

This last equation implies that if $a = 1$, then $\hat{a} = 1$, *i. e.* conservative scattering remains conservative under the scaling transformation.

Since expansion of the phase function in Legendre polynomials has been shown to be an extremely useful way of "isolating" the azimuth dependence in slab geometry, we proceed by expanding both phase functions in this manner

$$p(\cos\Theta) = \sum_{n=0}^{\infty}(2n+1)\chi_n P_n(\cos\Theta) \tag{J.12}$$

$$\hat{p}(\cos\Theta) = \sum_{n=0}^{\infty}(2n+1)\hat{\chi}_n P_n(\cos\Theta) \tag{J.13}$$

where $P_n(\cos\Theta)$ is the Legendre polynomial, and the expansion coefficients are defined by eqn. 6.26. The δ-function may also be expanded in Legendre polynomials, i. e.

$$4\pi\delta(\hat{\Omega}'-\hat{\Omega}) = 4\pi\delta(\mu'-\mu)\delta(\phi'-\phi) = 2\delta(1-\cos\Theta) = \sum_{n=0}^{\infty}(2n+1)P_n(\cos\Theta). \tag{J.14}$$

We note that the expansion coefficients in this case are all unity. Substitution of eqns. J.12 and J.14 into eqn. J.6 yields

$$\sum_{n=0}^{\infty}[\hat{a}\hat{\chi}_n - \beta a\chi_n - (1-\beta)](2n+1)P_n(\cos\Theta) = 0 \qquad \text{(J.15)}$$

which implies

$$\hat{a}\hat{\chi}_n = \beta a\chi_n + (1-\beta) \qquad \text{(J.16)}$$

or

$$1 - \hat{a}\hat{\chi}_n = \beta(1 - a\chi_n) \qquad \text{(J.17)}$$

or

$$\hat{\tau}\hat{a}(1-\hat{\chi}_n) = \tau a(1-\chi_n) \qquad \text{(J.18)}$$

where we have used eqns. J.3 and J.9 in the last step. Since $\chi_0 = 1$ eqns. J.16 and J.10 imply $\hat{a}\hat{\chi}_0 = \beta a + 1 - \beta = \hat{a}$ or $\hat{\chi}_0 = 1$. This shows that the expanded *scaled* phase function is correctly normalized as implied by eqn. J.10.

Finally, by defining $h_n = (2n+1)(1-a\chi_n)$ and using eqn. J.17, we obtain

$$\hat{h}_n = (2n+1)(1-\hat{a}\hat{\chi}_n) = \beta h_n = (\tau/\hat{\tau})h_n \qquad \text{(J.19)}$$

or

$$\hat{\tau}\hat{h}_n = \tau h_n. \qquad \text{(J.20)}$$

The radiative transfer equation is invariant under

scale changes of the optical depth

and phase function which leave invariant the parameter

$$\eta_n \equiv h_n\tau = (2n+1)(1-a\chi_n)\tau \qquad \text{(J.21)}$$

It is clear that $\beta = 1 - af$ in the $\delta - M$ method.