Appendix L

Spherical Shell Geometry

For solar zenith angles greater than about 80° and twilight situations, we have to take the curvature of the earth into account and solve the radiative transfer equation appropriate for a spherical shell atmosphere.† The geometry is illustrated in Figure AL.1.

In spherical shell geometry, the derivative of the intensity consists of three terms in addition to the one term occurring for slab geometry. These additional terms express the change in the intensity associated

† The treatment of spherical geometry is described in: V. V. Sobolev, *Light Scattering in Planetary Atmospheres* (Transl. by W. M. Irvine), Pergamon, 256 pp., 1975.

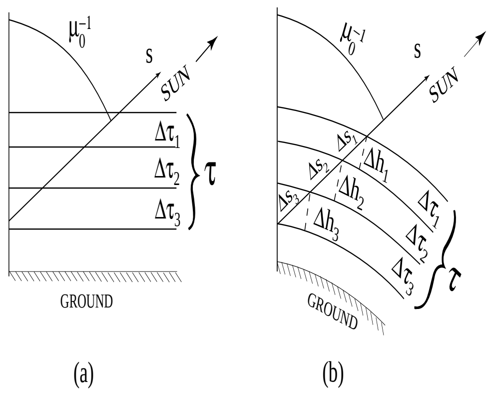

Fig. AL.1. Illustration of plane versus spherical geometry. (a) In plane geometry the slant path is the same for all layers of equal geometrical thickness. (b) In spherical geometry the slant path changes from layer to layer.

with changes in polar angle, azimuthal angle, and solar zenith angle. Hence, for a spherical shell medium illuminated by a direct (collimated) beam of radiation, the appropriate radiative transfer equation for the diffuse intensity may be expressed as (see §6.4)

$$\hat{\Omega} \cdot \nabla I(r, u, \phi, \mu_0) = -k(r)[I(r, u, \phi, \mu_0) - S(r, u, \phi, \mu_0)]. \tag{L.1}$$

Here r is the distance from the center of the planet and k is the extinction coefficient, while u and ϕ are the cosine of the polar angle and the azimuthal angle, respectively. The symbol $\hat{\Omega} \cdot \nabla$ denotes the derivative operator or the 'streaming term' appropriate for this geometry. To arrive at this term we must use spherical geometry. If we map the intensity from a set of global spherical coordinates to a local set with reference to the local zenith direction, then as explained in Appendix O, the streaming term becomes†

$$\begin{aligned} \hat{\Omega} \cdot \nabla \quad \equiv \quad & u\frac{\partial}{\partial r} + \frac{1-u^2}{r}\frac{\partial}{\partial u} \\ & + \frac{1}{r}f(u, \mu_0)\left[\cos(\phi - \phi_0)\frac{\partial}{\partial \mu_0} + \frac{\mu_0}{1-\mu_0^2}\sin(\phi-\phi_0)\frac{\partial}{\partial(\phi - \phi_0)}\right] \end{aligned} \tag{L.2}$$

where the factor f is given by

$$f(u, \mu_0) \equiv \sqrt{1-u^2}\sqrt{1-\mu_0^2}. \tag{L.3}$$

For slab geometry, only the first term contributes. The curvature gives rise to additional terms. Thus, for spherically symmetric geometry, the second term must be added, while the third and fourth terms are required for a spherical shell medium illuminated by direct (collimated) beam radiation. The source function in eqn. L.1 is

$$\begin{aligned} S(r, u, \phi, \mu_0) \quad \equiv \quad & \frac{a(r)}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' p(r, u', \phi'; u, \phi) I(r, u', \phi', \mu_0) \\ & + \frac{a(r)}{4\pi} p(r, -\mu_0, \phi_0; u, \phi) F^s e^{-\tau Ch(r, \mu_0)}. \end{aligned} \tag{L.4}$$

† The derivation of the 'streaming' term given in Appendix O is taken from: A. Kylling: *Radiation Transport in Cloudy and Aerosol Loaded Atmospheres*, Ph. D. Thesis, University of Alaska, Fairbanks, USA, 1992, and the discussion of the azimuthally-averaged equation from: A. Dahlback and K. Stamnes, *A new spherical model for computing the radiation field available for photolysis and heating at twilight*, Planet. Space Sci., **39**, 671–683, 1991.

The first term in eqn. L.4 is due to multiple scattering and the second term is due to first-order scattering. We have used the diffuse/direct splitting so that eqn. L.1 describes the diffuse radiation field only. We note that for isotropic scattering, the primary scattering 'driving term' becomes isotropic, which implies that the intensity becomes azimuth independent. The argument in the exponential, $Ch(r, \mu_0)$, is the *air-mass factor* or the *Chapman function*: the quantity by which the vertical optical depth must be multiplied to obtain the slant optical path. For a slab geometry, $Ch(r, \mu_0) = 1/\mu_0 = \sec\theta_0$. Other properties of $Ch(r, \mu_0)$ are explored in Problems L.1 and L.2. Hence $\exp[-\tau Ch(r, \mu_0)]$ yields the attenuation of the incident solar radiation of flux F^s (normal to the beam) along the solar beam path.

We find that eqn. L.4 may be written as follows

$$S(r, u, \phi, \mu_0) = \frac{a(r)}{4\pi} \int_0^{2\pi} d\phi' \int_{-1}^{1} du' \left[\sum_{m=0}^{2N-1} (2 - \delta_{0m}) p^m(\tau, u', u) \cos m(\phi - \phi') \right] I(r, u', \phi') + \left[\sum_{m=0}^{2N-1} X_0^m(\tau, u) \cos m(\phi - \phi_0) \right] e^{-\tau Ch(r, \mu_0)} \tag{L.5}$$

where $p^m(\tau, u', u)$ and $X_0^m(\tau, u)$ are defined by eqns. 6.33 and 6.36.

AL.1 "Isolation" of Azimuth Dependence

The extra derivative terms in eqn. L.2 makes the spherical geometry case more difficult to treat than the corresponding slab problem. In general, we could expand the intensity in a Fourier series containing both *sine* and em cosine terms to account for the appearence of both types of terms in the derivative operator. However, if the effects of sphericity are small, it is useful to treat the second, third, and fourth derivative terms in eqn. L.2 (which are due to the spherical geometry) as a perturbation. Thus, if we ignore these terms, we are left with a plane parallel problem to solve and the derivative terms can be included in an iterative manner by utilizing the plane parallel solutions. Then, since the first term in eqn. L.5 is essentially a Fourier cosine series, and the diffuse intensity described by eqn. L.1 is driven by the second term in eqn. L.5, which contains only *cosine* terms, we may expand the intensity as previously expressed by eqn. 6.34 ignoring *sine* terms.

This is because we have assumed that the third and fourth terms in eqn. L.2, which contain *sine* terms, can be treated as a perturbation and hence can be evaluated in an iterative manner from the plane parallel solutions.

With these assumptions, eqn. L.1 becomes

$$\sum_{m=0}^{2N-1}\left\{u\frac{\partial I^m(r,u,\mu_0)}{\partial r}+\frac{1-\mu_0^2}{r}\frac{\partial I^m}{\partial u}\right.$$
$$\left.+\,k(r)\left[I^m(r,u,\mu_0)-S^m(r,u,\mu_0)\right]\right\}\cos m(\phi_0-\phi)=J(r,u,\phi,\mu_0). \tag{L.6}$$

Here

$$S^m(r,u,\mu_0)\equiv\frac{a(r)}{2}\int_{-1}^{1}p^m(r,u',u)I^m(r,u')du'+X_0^m(r,u)e^{-\tau Ch(r,\mu_0)} \tag{L.7}$$

and

$$J(r,u,\phi,\mu_0)\equiv\frac{1}{r}f(u,\mu_0)\left\{\cos(\phi_0-\phi)\sum_{m=0}^{2N-1}\cos m(\phi_0-\phi)\frac{\partial I^m(r,u,\mu_0)}{\partial\mu_0}\right.$$
$$\left.+\frac{\mu_0}{1-\mu_0^2}\sin(\phi-\phi_0)\sum_{m=0}^{2N-1}m\sin m(\phi-\phi_0)I^m(r,u,\mu_0)\right\}. \tag{L.8}$$

In the following example, we describe how the equations may be solved in a simplified geometry.

Example: Zenith Sky and Mean Intensity – Iterative Approach

If we are interested in only the zenith sky intensity (which is azimuthally independent), then only the $m = 0$ term in eqn. 6.34 contributes. For $m = 0$, the second term in eqn. 6.34 is identically zero. Upon averaging over azimuth the first term becomes proportional to $\partial I^1(r,u,\mu_0)/\partial\mu_0$ and may also be discarded if our interest lies solely in the zenith sky intensity. Thus, the zenith sky intensity is obtained by setting $J(r,u,\mu_0) = 0$ in eqn. L.6 and solving it for $m = 0$ only. Similarly, for isotropic scattering there is no azimuth dependence and the complete solution is again arrived at by setting $J(r,u,\mu_0) = 0$ in eqn. L.6 and solving the equation for $m = 0$ only.

If our interest is in photolysis and heating rates, only the mean intensity is needed. We therefore average eqn. L.6 over azimuth to obtain (see also Appendix O):

$$u\frac{\partial I^0(r,u,\mu_0)}{\partial r}+\frac{1-\mu_0^2}{r}\frac{\partial I^0}{\partial u}+\frac{1}{r}\left[J_1(r,u,\mu_0|I^1)+J_2(r,u,\mu_0|I^1)\right]=-k(r)\left[I^0(r,u,\mu_0)-S^0(r,u,\mu_0)\right]$$

where $S^0(r, u, \mu_0)$ is obtained by setting $m = 0$ in eqn. L.7 and

$$J_1(r, u, \mu_0 | I^1) = \frac{1}{2} f(u, \mu_0) \frac{\partial I^1(r, u, \mu_0)}{\partial \mu_0}$$

$$J_2(r, u, \mu_0 | I^1) = \frac{1}{2} f(u, \mu_0) \frac{\mu_0}{1 - \mu_0^2} I^1(r, u, \mu_0).$$

We note that J_1 and J_2 depend functionally on the first azimuth-dependent Fourier component of the intensity, I^1, as indicated. Dividing by $-k(r)$, and introducing $d\tau = -k(r)dr$, we obtain

$$u \frac{\partial I(\tau, u)}{\partial \tau} = I(\tau, u) - \frac{a(r)}{2} \int_{-1}^{1} du' p(r, u', u) I(r, u') - S^*(\tau, u)$$

where

$$S^*(\tau, u) \equiv X_0(\tau(r), u) e^{-\tau Ch[\tau, \mu_0]} + \frac{1 - u^2}{kr} \frac{\partial I}{\partial u} + \frac{1}{kr}(J_1 + J_2). \tag{L.9}$$

To simplify the notation, we have dropped the $m = 0$ superscript. If we ignore the three last terms in the expression for $S^*(\tau, u)$, we are left with an equation which is identical to that obtained for plane geometry except that the primary scattering term is evaluated in spherical geometry using the correct path length. We shall refer to this approach, in which the primary scattering driving term is included correctly but the multiple scattering is done in plane geometry, as the 'pseudo-spherical' approximation. Having obtained a 'pseudo-spherical' solution, we may proceed to evaluate the terms we neglected and then solve the equation again including those terms. Repetition of this procedure provides an iteration scheme that is expected to converge if the perturbation terms (i.e., the three last terms on the right side of eqn. L.9) are small compared with the driving term. We shall provide an example of this approach later in the book. Suffice it to say here that this approach has been found to be quite useful for obtaining both the mean intensity and the zenith sky intensity in twilight situations.

In a stratified planetary atmosphere, spherical effects (i. e., the angle derivatives), become important around sunrise and sunset. Thus, the first term in eqn. L.9 is the dominant one and the other terms may be treated as perturbations. It has been shown (by using a perturbation technique to account for the spherical effects) that in a stratified atmosphere, mean intensities may be calculated with sufficient accuracy for zenith angles less than 90° by including only the first term in eqn. L.9, when spherical geometry is used to compute the direct beam attenuation. Then, we may ignore all angle derivatives and simply write the streaming term as

$$\hat{\Omega} \cdot \nabla \cong uk \frac{\partial}{\partial \tau}. \tag{L.10}$$

While this 'pseudo-spherical' approach works adequately for the computation of intensities in the zenith– and nadir-viewing directions, and mean intensities (for zenith angles less than 90°), it may not work for computation of intensities in directions off-zenith (or off-nadir) unless it can be shown that the angle derivative terms are indeed small.

AL.2 Problems

1. The optical depth in a curved atmosphere is required to compute the attenuation of solar irradiance. For an overhead sun, the vertical optical depth between altitude z_0 and the sun is

$$\tau(z_0, \nu) = \int_{z_0}^{\infty} dz k(z, \nu)$$

where $k(z, \nu)$ is the extinction coefficient at frequency ν, and dz is measured along the vertical. For a non-vertical path dz must be replaced by the actual length along the ray path. In slab geometry the actual path length along a ray is simply dz/μ_0 where μ_0 is the cosine of the solar zenith angle. In spherical geometry the situation is somewhat more complex. Then dz must be replaced by the actual ray path through a curved atmosphere.

(a) For solar zenith angles $\theta_0 < 90°$, use geometrical considerations to derive the following expression for the optical depth between level z_0 and the sun in a spherical atmosphere

$$\tau(z_0, \nu, \mu_0) = \int_{z_0}^{\infty} dz \frac{k(z, \nu)}{\sqrt{1 - \left(\frac{R+z_0}{R+z}\right)^2 (1 - \mu_0^2)}} \qquad (\theta_0 < 90°)$$

where R is the radius of the planet and z_0 the distance above the Earth's surface.

(b) Similarly for $\theta_0 > 90°$ show that the following expression applies

$$\begin{aligned} \tau(z_0, \nu, \mu_0) &= 2\int_{z_s}^{\infty} dz k(z, \nu) \left[1 - \left(\frac{R + z_s}{R + z}\right)^2\right]^{-\frac{1}{2}} \\ &\quad - \int_{z_0}^{\infty} dz k(z, \nu) \left[1 - \left(\frac{R + z_0}{R + z}\right)^2 (1 - \mu_0^2)\right]^{-\frac{1}{2}} \end{aligned}$$

where z_s is a screening height below which the atmosphere is essentially opaque to radiation of frequency ν.

For practical computations we may divide the spherical atmosphere into a number of concentric shells. Let Δh_j denote the (vertical) thickness of the shell lying between r_j ($r_j = R + z_j$) and r_{j+1} ($r_{j+1} = r_j - \Delta h_j$)

where z_j is the vertical distance from the surface of the planet to location r_j. (Note that r_1 is at the top of the atmosphere and r_{L+1} is at the bottom of the deepest layer (shell) considered if the atmosphere is divided into L concentric shells.)

(c) Show that approximate expressions for the optical depth that may be used in practical computations are given by

$$\tau(\tau, \nu, \mu_0) = \sum_{j=1}^{p} \Delta\tau_j^v \left(\frac{\Delta S_j}{\Delta h_j}\right) \qquad \theta_0 < 90^\circ$$

$$\tau(\tau, \nu, \mu_0) = \sum_{j=1}^{p} \Delta\tau_j \left(\frac{\Delta S_j}{\Delta h_j}\right) + 2 \sum_{j=p+1}^{L-1} \Delta\tau_j \left(\frac{\Delta S_j}{\Delta h_j}\right) + \Delta\tau_L \left(\frac{\Delta S_L}{\Delta h_L}\right) \qquad (\theta_0 > 90^\circ).$$

Here L is the layer in the atmosphere below which attenuation is complete, τ_j is the vertical optical depth of shell j, and

$$\Delta S_j = \sqrt{r_j^2 - r_p^2(1-\mu_0^2)} - \sqrt{r_{j+1}^2 - r_p^2(1-\mu_0^2)}$$

where r_j and r_{j+1} are the distances from the center of the planet to the upper and lower boundary, respectively of layer j, and r_p is the distance from the center to the point at which the optical depth is evaluated.

2. (a) Show that the Chapman function may be written

$$Ch(X, \theta) \equiv \frac{\mathcal{N}(z, \theta)}{n(z)H} = \int_0^\infty dY \exp[-\sqrt{X^2 + 2XY\cos\theta + Y^2} + X].$$

Here $X = R_\oplus/H$, $Y = z/H$, and $\mathcal{N}(z, \theta)$ is the slant column number for a spherically-symmetric exponential atmosphere. (b) Defining $\ln V = -\sqrt{X^2 + 2XY\cos\theta + Y^2} + X$, show that

$$Ch(X, \theta) = \int_0^1 \frac{dV(1 - \ln V/X)}{\sqrt{(1 + \sin\theta - \frac{\ln V}{X})(1 - \sin\theta - \frac{\ln V}{X})}}.$$

(c) Using the relationship

$$\int_0^1 \frac{dV}{\sqrt{\xi^2 - \ln V}} = 2e^{\xi^2} \int_\xi^\infty ds e^{-s^2}$$

show that, on neglecting terms of order X^{-1},

$$Ch(X, \theta) = \sqrt{2X} e^{X \cos^2 \theta/2} [1 - erf(\sqrt{X/2} \cos \theta)]$$

where erf is the error function.

(c) Show that, to order X^{-2}, that

$$Ch(X, \theta) = \frac{2\xi e^{\xi^2}}{\cos \theta} [1 - erf(\xi)]$$

where $\xi = \sqrt{X/2} \cot \theta$.

(e) Show that $Ch(X \to \infty, \theta) \to \sec \theta$ for both forms (c) and (d).