Appendix M

Reciprocity for the Bidirectional Reflectance

In this appendix we prove the *Principle of Reciprocity* for the bidirectional reflectance, that is

$$\rho(\nu;\theta',\phi';\theta,\phi) = \rho(\nu;\theta,\phi;\theta',\phi'). \tag{M.1}$$

Referring to Fig. M.1, the proof first determines the exchange of radiative energy between the black elements dA_1 and dA_2 in a *hohlraum* due to reflection of energy by the surface dA_2. The theorem is proven by equating the energy exchange $dE_{\nu 123}$ from 1 to 3 via 2, and the energy

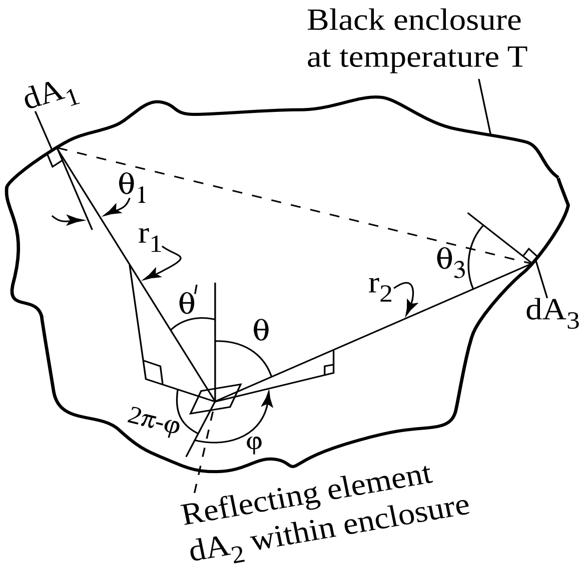

Fig. AM.1. Exchange of radiative energy within a *hohlraum*.

exchange $dE_{\nu 321}$ from 3 to 1 via 2, and equating these two quantities (see Figure AM.1).

The energy exchange $dE_{\nu 123}$ must balance the reciprocal energy exchange $dE_{\nu 321}$ in a TE situation. Otherwise there would be a net heating/cooling of one of the areas at the expense of the other, and this violates the conditions of the *hohlraum*. Let's first consider the exchange from 1 to 3 via 2. The radiative energy reflected by dA_2 and intercepted by dA_3 is given by that 'emitted' into the solid angle subtended by dA_3, $dA_3 \cos\theta_3 / r_2^2$,

$$dE_{\nu 123} = I^+_{\nu r2}(\theta', \phi'; \theta, \phi) \cos\theta \frac{dA_3 \cos\theta_3}{r_2^2} d\nu dt. \tag{M.2}$$

The reflected intensity $I^+_{\nu r2}$ is related to the intensity $I^-_\nu(\theta', \phi')$ arriving at dA_2 through eqn. 5.15

$$I^-_{\nu r2}(\theta', \phi') = \cos\theta' \rho(\nu; \theta', \phi'; \theta, \phi) I^-_{\nu 1}(\theta', \phi') d\omega_{21} \tag{M.3}$$

where

$$d\omega_{21} = \frac{dA_1}{r_1^2} \cos\theta_1 \tag{M.4}$$

is the angle subtended by dA_1 from the point dA_2. Putting these together, we find that the rate of energy exchange per unit frequency from 1 to 3 via 2 is

$$\frac{dE_{\nu 123}}{d\nu dt} = I^-_{\nu 1}(\theta', \phi') \rho(\nu; \theta', \phi'; \theta, \phi) \cos\theta' \left[\frac{dA_1}{r_1^2} \cos\theta_1\right] \cos\theta \left[\frac{dA_3}{r_2^2} \cos\theta_3\right]. \tag{M.5}$$

Now consider energy exchange in the reverse direction, 3 to 1 via 2. The rate of energy per unit frequency reflected at dA_2 into the direction of dA_1 is

$$\frac{dE_{\nu 321}}{d\nu dt} = I^+_{\nu r2}(\theta, \phi; \theta', \phi') \cos\theta' dA_2 \left[\frac{dA_1}{r_1^2} \cos\theta_1\right]. \tag{M.6}$$

But the reflected intensity at dA_2 is given by

$$I^+_{\nu r2}(\theta, \phi; \theta', \phi') = I^-_{\nu 3}(\theta, \phi)) \rho(\nu; \theta, \phi; \theta', \phi') \cos\theta d\omega_{23} \tag{M.7}$$

where $d\omega_{23} = dA_3 \cos\theta_3 / r_2^2$ is the solid angle subtended by dA_3 at the point dA_2. Putting these together, we find for the energy exchange rate

$$\frac{dE_{\nu 321}}{d\nu dt} = I^-_{\nu 3}(\theta, \phi) \rho(\nu; \theta, \phi; \theta', \phi') \cos\theta' \cos\theta \left[\frac{dA_3}{r_2^2} \cos\theta_3\right] \left[\frac{dA_1}{r_1^2} \cos\theta_1\right]. \tag{M.8}$$

Equating the two rates of energy exchange, we find

$$I_{\nu 1}^{-}(\theta', \phi')\rho(\nu; \theta', \phi'; \theta, \phi) = I_{\nu 3}^{-}(\theta, \phi)\rho(\nu; \theta, \phi; \theta', \phi'). \tag{M.9}$$

But in TE, the two intensities are just the Planck function, $I_{\nu 1}^{-} = I_{\nu 3}^{-} = B_\nu$. Therefore

$$\rho(\nu; \theta', \phi'; \theta, \phi) = \rho(\nu; \theta, \phi; \theta', \phi'). \tag{M.10}$$

Given the above reciprocity property for the BDRF, we now show that reciprocity also applies to the flux reflectance. Placing the two definitions together, we have

$$\rho(\nu; -\hat{\Omega}', 2\pi) = \int_{+} d\omega \cos\theta \rho(\nu; -\hat{\Omega}', \hat{\Omega}) \tag{M.11}$$

$$\rho(\nu; 2\pi, \hat{\Omega}) = \int_{-} d\omega' \cos\theta' \rho(\nu; -\hat{\Omega}', \hat{\Omega}). \tag{M.12}$$

The first quantity, $\rho(\nu; -\hat{\Omega}', 2\pi)$ is the *directional-hemispherical reflectance*, and the second quantity, $\rho(\nu; 2\pi, \hat{\Omega})$ is the *hemispherical-directional reflectance.* If we evaluate the first of the above equations at $\hat{\Omega}' = \hat{\Omega}$, and place primes on the angular integration variables (realizing that they are dummy variables), we have

$$\rho(\nu; -\hat{\Omega}, 2\pi) = \int_{+} d\omega' \cos\theta' \rho(\nu; -\hat{\Omega}, \hat{\Omega}'). \tag{M.13}$$

Invoking reciprocity of the BDRF, $\rho(\nu; -\hat{\Omega}, \hat{\Omega}') = \rho(\nu; -\hat{\Omega}', \hat{\Omega})$, we have

$$\rho(\nu; -\hat{\Omega}, 2\pi) = \int_{+} d\omega' \cos\theta' \rho(\nu; -\hat{\Omega}', \hat{\Omega}). \tag{M.14}$$

But this is the same expression for the hemispherical-directional reflectance, eqn. M.12. Thus we find the desired reciprocity relationship

$$\rho(\nu; -\hat{\Omega}, 2\pi) = \rho(\nu; 2\pi; \hat{\Omega}). \tag{M.15}$$