Appendix N

Isolation of the Azimuth-Dependence

The purpose of this Appendix is to provide a derivation of the azimuthal components of the intensity field. We start with the half-range equations for the diffuse intensity which we write in full-range form for the present purpose

$$u\frac{dI(\tau,u,\phi)}{d\tau} = \\ I(\tau,u,\phi) - \frac{a}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} \left\{ \mathrm{d}u' p(u',\phi';u,\phi) I(\tau,u',\phi') \\ -\frac{aF^s}{4\pi} p(-\mu_0,\phi_0;u,\phi)e^{-\tau/\mu_0} \right\}. \tag{N.1}$$

Since, as noted in §6.3 the expansion of the phase function in Legendre polynomials is essentially a Fourier cosine series, i.e.

$$p(u',\phi';u,\phi) = \sum_{m=0}^{2N-1} (2-\delta_{0m})\, p^m(u',u)\cos[m(\phi-\phi')], \tag{N.2}$$

where

$$p^m(u',u) = \sum_{l=m}^{2N-1} (2l+1)\chi_l^m \Lambda_l^m(u)\Lambda_l^m(u') \tag{N.3}$$

we expand the intensity likewise

$$I(\tau,u,\phi) = \sum_{m=0}^{2N-1} I^m(\tau,u)\cos[m(\phi_0-\phi)]. \tag{N.4}$$

Substitution of eqns. N.2 and N.4 into the integral term of eqn. N.1 yields

$$\frac{a}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' p(u',\phi';u,\phi) I(\tau,u',\phi') =$$

$$\frac{a}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' \left\{ \sum_{m=0}^{2N-1} (2-\delta_{0m})\, p^m(u',u)\cos[m(\phi-\phi')] \right\} \cdot \left\{ \sum_{r=0}^{2N-1} I^r(\tau,u')\cos[r(\phi_0-\phi')] \right\}. \tag{N.5}$$

Focussing on the integration over azimuth we find that for arbitrary m-values only the $r = m$ term contributes. Thus, we obtain $2\pi I^0(\tau,u')$ for $m = 0$, $2\pi I^1(\tau,u') cos(\phi_0 - \phi)$ for $m = 1$, and in general

$$\sum_{m=0}^{2N-1} \int_0^{2\pi} d\phi' (2-\delta_{0m}) \sum_{r=0}^{2N-1} I^r(\tau,u')\cos[m(\phi-\phi')]\cos[r(\phi_0-\phi')] = 2\pi \sum_{m=0}^{2N-1} I^m(\tau,u')\cos[m(\phi_0-\phi)]. \tag{N.6}$$

Therefore eqn. N.5 reduces to

$$\frac{a}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' p(u',\phi';u,\phi) I(\tau,u',\phi') = \sum_{m=0}^{2N-1} (2-\delta_{0m}) \left\{ \frac{a}{2}\int_{-1}^{1} du' p^m(u',u) I^m(\tau,u') \right\} \cos[m(\phi_0-\phi)]. \tag{N.7}$$

It is now clear that substitution of eqns. N.2 and N.4 into eqn. N.1 yields the desired result given in Chapter 6, i.e. eqns. 6.33–6.36.

AN.1 Treatment of the Lower Boundary Condition

Since we are dealing with reflection it is natural to use half-range quantities here. The diffuse reflectance at the lower boundary, $\tau = \tau^*$, is written as (see §6.9.4)

$$I^+(\tau^*,\mu,\phi) = \epsilon(\mu)B(T_s) + \frac{\mu_0 F^s}{\pi}\rho_d(-\mu_0,\phi_0;\mu,\phi)e^{-\tau^*/\mu_0} + \frac{1}{\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} d\mu' \mu' \rho_d(-\mu',\phi';\mu,\phi) I^-(\tau^*,\mu',\phi') \tag{N.8}$$

where ρ_d is the bidirectional reflectance and ϵ is the emittance. First we note that only the $m = 0$ component of the intensity contributes to fluxes, since

$$\begin{aligned} F^{\pm} &= \int_0^{2\pi} d\phi \int_0^1 d\mu\mu I^{\pm}(\tau,\mu,\phi) \\ &= \int_0^{2\pi} d\phi \int_0^1 d\mu\mu \sum_{m=0}^{2N-1} I^{m\pm}(\tau,\mu)\cos[m(\phi-\phi_0)] \\ &= 2\pi \int_0^1 d\mu\mu I^{0\pm}(\tau,\mu). \end{aligned} \tag{N.9}$$

Next we note that Kirchhoff's law states

$$\epsilon(\mu) + \frac{1}{\pi}\int_0^{2\pi} d\phi' \int_0^1 d\mu'\mu'\rho_d(-\mu',\phi';\mu,\phi) = 1 \tag{N.10}$$

suggesting that we should use eqn. N.10 to compute the emittance from the reflectance for consistency. Below we shall start by looking at the simple case of a Lambert reflector before we consider the more general case.

AN.2 Lambertian Surface

A Lambert reflector is defined such that the reflected radiation is isotropic regardless of the directional dependence of the incident radiation. This implies that the bidirectional reflectance is independent of direction, i. e., $\rho_d(-\mu',\phi';\mu,\phi) = \rho_L = \textit{constant}$. Now, integrating the left side of eqn. N.8, we find that the reflected flux becomes

$$F^+(\tau^*) = \int_0^{2\pi} d\phi \int_0^1 d\mu\mu I^+(\tau^*,\mu,\phi) = \pi I^{0+}(\tau^*) \tag{N.11}$$

since the reflected radiation is isotropic. Integration of the first term on the right side yields $\pi\epsilon B(T_s)$, where we have used Kirchhoff's law yielding $\epsilon(\mu) + \rho_L = 1$, which implies ϵ=constant (independent of μ) in this special case. The second term yields $\rho_L\mu_0 F^s e^{-\tau^*/\mu_0}$, and the third term becomes

$$\begin{aligned} \int_0^{2\pi} d\phi \int_0^1 \mu d\mu \left[\frac{\rho_L}{\pi}\int_0^{2\pi} d\phi' \int_0^1 d\mu'\mu' I^-(\tau^*,\mu',\phi')\right] &= \\ 2\pi\rho_L \int_0^1 d\mu'\mu' I^{0-}(\tau^*,\mu') & \end{aligned} \tag{N.12}$$

where $I^{0-}(\tau^*,\mu') = \frac{1}{2\pi}\int_0^{2\pi} I^-(\tau^*,\mu',\phi)d\phi$ is the azimuthally-averaged downward intensity (or the $m = 0$ azimuthal component since we have expressed the intensity in a Fourier cosine series). Thus, for a Lambert reflector we have the following simple boundary condition relating the intensity reflected by the surface to the downward intensity there

$$I^{0+}(\tau^*) = \epsilon B(T_s) + \frac{\mu_0}{\pi} F^s \rho_L e^{-\tau^*/\mu_0} + 2\rho_L \int_0^1 d\mu' \mu' I^{0-}(\tau^*, \mu'). \qquad \text{(N.13)}$$

AN.3 Non-Lambertian Surface

We shall assume that the bidirectional reflectance is azimuthally-symmetric so that we may expand it in a Fourier cosine series as

$$\rho_d(-\mu', \phi'; \mu, \phi) = \sum_{m=0}^{2N-1} \rho_d^m(-\mu', \mu) \cos[m(\phi' - \phi)]. \qquad \text{(N.14)}$$

In this more general case we find that the third term on the right side of eqn. N.8 becomes

$$\frac{1}{\pi}\int_0^{2\pi} d\phi' \int_{-1}^1 d\mu' \mu' \rho_d(-\mu', \phi'; \mu, \phi) I^-(\tau^*, \mu', \phi') = \frac{1}{\pi}\int_0^{2\pi} d\phi' \int_{-1}^1 d\mu' \mu' \left\{ \sum_{m=0}^{2N-1} \rho^m(-\mu', \mu) \cos[m(\phi' - \phi)] \cdot \sum_{r=0}^{2N-1} I^{r-}(\tau^*, \mu') \cos[r(\phi_0 - \phi')] \right\}. \qquad \text{(N.15)}$$

Since

$$\sum_{m=0}^{2N-1} \int_0^{2\pi} d\phi' \sum_{r=0}^{2N-1} I^{r-}(\tau^*, \mu') \cos[m(\phi' - \phi)] \cos[r(\phi_0 - \phi')] = \pi(1 + \delta_{0m}) \sum_{m=0}^{2N-1} I^{m-}(\tau^*, \mu') \cos[m(\phi_0 - \phi)] \qquad \text{(N.16)}$$

we find

$$\frac{1}{\pi}\int_0^{2\pi} d\phi' \int_{-1}^1 d\mu' \mu' \rho_d(-\mu', \phi'; \mu, \phi) I^-(\tau^*, \mu', \phi') = \sum_{m=0}^{2N-1} \left\{ 2\int_0^1 d\mu' \mu' \rho^m(-\mu', \mu) I^{m-}(\tau^*, \mu') \right\} \cos[m(\phi_0 - \phi)]. \qquad \text{(N.17)}$$

Finally, substitution of eqns. N.4 and N.17 into N.8 yields

$$\sum_{m=0}^{2N-1} \left\{ I^{m+}(\tau^*, \mu) - \epsilon(\mu) B(T_s) \delta_{0m} - \frac{1}{\pi} F^s \rho_d^m(-\mu_0, \mu) e^{-\tau^*/\mu_0} - \right.$$
$$\left. (1 + \delta_{0m}) \int_0^1 d\mu' \mu' \rho_d^m(-\mu', \mu) I^{m-}(\tau^*, \mu') \right\} \cos[m(\phi_0 - \phi)] = 0. \quad \text{(N.18)}$$

Thus, we see that each Fourier component of the intensity must satisfy the boundary condition

$$\begin{aligned} I^{m+}(\tau^*, \mu) &= \epsilon(\mu) B(T_s) \delta_{0m} + \frac{1}{\pi} F^s \rho_d^m(-\mu_0, \mu) e^{-\tau^*/\mu_0} \\ &\quad + (1 + \delta_{0m}) \int_0^1 d\mu' \mu' \rho_d^m(-\mu', \mu) I^{m-}(\tau^*, \mu'). \quad \text{(N.19)} \end{aligned}$$

We note that for $m = 0$ and $\rho_d = constant = \rho_L$ we retain the azimuthally-independent case pertinent for a Lambertian surface considered above as we should.