Appendix O

The Streaming term in Spherical Geometry

Since the Earth's atmosphere has the form of a spherical shell, the radiative transfer equation must be cast in a form applicable to spherical geometry. The components of the streaming term $(\hat{\Omega} \cdot \nabla)$ in spherical geometry are

$$
\begin{aligned}
\hat{\Omega} &= \cos\Phi \sin\Theta \; \mathbf{e}_x + \sin\Phi \sin\Theta \; \mathbf{e}_y + \cos\Theta \; \mathbf{e}_z \qquad (O.1) \\
\nabla &= \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_{\Theta_0} \frac{1}{r} \frac{\partial}{\partial \Theta_0} + \mathbf{e}_{\Phi_0} \frac{1}{r \sin\Theta_0} \frac{\partial}{\partial \Phi_0}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{e}_r &= \sin\Theta_0 \cos\Phi_0 \; \mathbf{e}_x + \sin\Theta_0 \sin\Phi_0 \; \mathbf{e}_y + \cos\Theta_0 \; \mathbf{e}_z \\
\mathbf{e}_\Theta &= \cos\Theta_0 \cos\Phi_0 \; \mathbf{e}_x + \cos\Theta_0 \sin\Phi_0 \; \mathbf{e}_y - \sin\Theta_0 \; \mathbf{e}_z \\
\mathbf{e}_\Phi &= -\sin\Phi_0 \; \mathbf{e}_x + \cos\Phi_0 \; \mathbf{e}_y
\end{aligned}
$$

and the angles are defined in Fig AO.1.

Taking the dot product of $\hat{\Omega}$ and ∇ gives

$$
\begin{aligned}
\hat{\Omega} \cdot \nabla &= [\cos\Theta \cos\Theta_0 + \sin\Theta \sin\Theta_0 \cos(\Phi_0 - \Phi)] \frac{\partial}{\partial r} \\
&\quad - \frac{1}{r} [\cos\Theta \sin\Theta_0 - \sin\Theta \cos\Theta_0 \cos(\Phi_0 - \Phi)] \frac{\partial}{\partial \Theta_0} \\
&\quad - \frac{1}{r} \frac{\sin\Theta}{\sin\Theta_0} \sin(\Phi_0 - \Phi) \frac{\partial}{\partial \Phi_0}. \qquad (O.2)
\end{aligned}
$$

For practical reasons it is preferable to refer the system of spherical coordinates to the local zenith direction. Thus we want to map the intensity from the set of global coordinates $(r, \Theta_0, \Phi_0, \Theta, \Phi)$ to the local

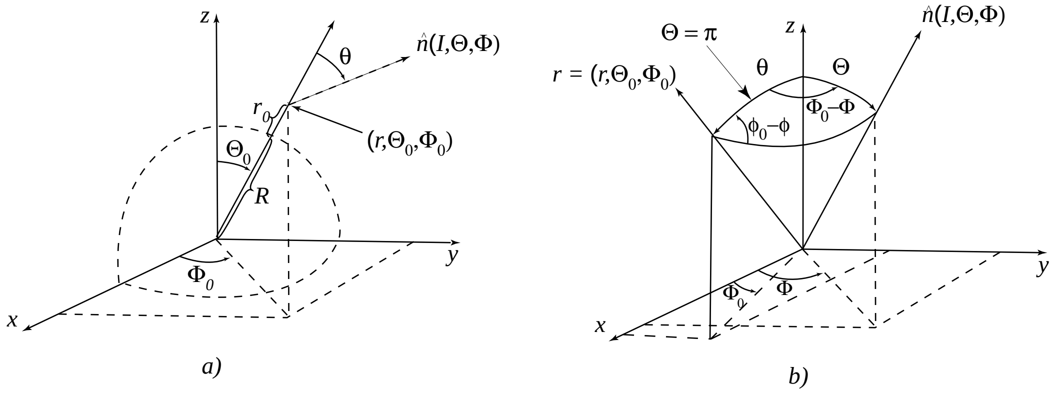

Fig. AO.1. The geometric setting. Note that in panel b the directional vector Ω has been parallel shifted to have its starting point at the surface of the earth.

set $(r, \mu_0, \phi_0, \mu, \phi)$, i.e.†

$$I(r, \Theta_0, \Phi_0, \Theta, \Phi) \;\Rightarrow\; I(r, \mu_0, \phi_0, \mu, \phi) \tag{O.3}$$

where

$$\mu \;\equiv\; \cos\theta \equiv \mathbf{e}_r \cdot \hat{\Omega} = \cos\Theta\cos\Theta_0 + \sin\Theta\sin\Theta_0\cos(\Phi_0 - \Phi) \tag{O.4}$$

$$\mu_0 \;\equiv\; \cos\theta_0 \tag{O.5}$$

and the local polar (θ_0, θ) and azimuthal angles (ϕ_0, ϕ) are defined in Fig. O.1. In view of eqn. O.4 we may rewrite O.2 as

$$\hat{\Omega}\cdot\nabla = \mu\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial\mu}{\partial\Theta_0}\frac{\partial}{\partial\Theta_0} + \frac{1}{r\,\sin^2\Theta_0}\frac{\partial\mu}{\partial\Phi_0}\frac{\partial}{\partial\Phi_0}. \tag{O.6}$$

Since μ is a function of both Θ_0 and Φ_0

$$\begin{aligned}\frac{\partial}{\partial\Theta_0} &= \frac{\partial}{\partial\theta_0} + \frac{\partial\mu}{\partial\Theta_0}\frac{\partial}{\partial\mu} \\ \frac{\partial}{\partial\Phi_0} &= \frac{\partial\phi_0}{\partial\Phi_0}\frac{\partial}{\partial\phi_0} + \frac{\partial\mu}{\partial\Phi_0}\frac{\partial}{\partial\mu}\end{aligned} \tag{O.7}$$

and eqn. O.6 becomes

$$\hat{\Omega}\cdot\nabla \;=\; \mu\frac{\partial}{\partial r} + \frac{1}{r}\left[\left(\frac{\partial\mu}{\partial\Theta_0}\right)^2 + \frac{1}{\sin^2\theta_0}\left(\frac{\partial\mu}{\partial\Phi_0}\right)^2\right]\frac{\partial}{\partial\mu} + \frac{1}{r}\frac{\partial\mu}{\partial\Theta_0}\frac{\partial}{\partial\theta_0}$$

† The global coordinates r, Θ_0 and Φ_0 denote a point in $\mathbf{R^3}$, whereas Θ and Φ are the coordinates of a point on the unit sphere $\mathbf{S^2} = \{x, y : x^2 + y^2 = 1\}$, and similar for the local coordinates. Hence both $I(r, \Theta_0, \Phi_0, \Theta, \Phi)$ and $I(r, \mu_0, \phi_0, \mu, \phi)$ are real–valued functions defined on $\mathbf{R^3}\times\mathbf{S^2}$.

$$+ \frac{1}{r \sin^2\theta_0} \frac{\partial\mu}{\partial\Phi_0} \frac{\partial\phi_0}{\partial\Phi_0} \frac{\partial}{\partial\phi_0}. \tag{O.8}$$

Using eqn. O.4 and some relationships from spherical trigonometry

$$\left[\left(\frac{\partial\mu}{\partial\Theta_0} \right)^2 + \frac{1}{\sin^2\theta_0} \left(\frac{\partial\mu}{\partial\Phi_0} \right)^2 \right] = 1 - \mu^2 \tag{O.9}$$

$$\frac{\partial\mu}{\partial\Theta_0} = -\cos\Theta\sin\Theta_0 + \sin\Theta\cos\Theta_0\cos(\Phi_0 - \Phi) = -\sqrt{1-\mu^2}\cos(\phi_0 - \phi) \tag{O.10}$$

$$\frac{\partial\mu}{\partial\Phi_0} = -\sin\Theta\sin\Theta_0\sin(\Phi_0 - \Phi) = -\sqrt{1-\mu^2}\sin\theta_0\sin(\phi_0 - \phi) \tag{O.11}$$

$$\frac{\partial\phi_0}{\partial\Phi_0} = \frac{\partial(\phi_0 - \phi)}{\partial(\Phi_0 - \Phi)} = \cos\theta_0\sin(\phi_0 - \phi) \tag{O.12}$$

we may finally write the streaming term in spherical geometry referenced to the local zenith direction as

$$\mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu} + \frac{\sqrt{1-\mu^2}\sqrt{1-\mu_0{}^2}}{r} \left[\cos(\phi - \phi_0)\frac{\partial}{\partial\mu_0} + \frac{\mu_0}{1-\mu_0^2}\sin(\phi - \phi_0)\frac{\partial}{\partial(\phi - \phi_0)} \right]. \tag{O.13}$$

We note that in plane–parallel geometry only the first term in eqn. O.13 is included. For a spherically symmetric atmosphere the second term must be added. The full expression is, as stated above, valid for an inhomogeneous spherical shell, i.e. a planetary atmosphere.

AO.1 The streaming term pertinent to calculation of mean intensities

Quite generally the intensity may be expanded in a Fourier series

$$I(r, \mu_0, \phi_0, \mu, \phi) = \sum_{m=0}^{\infty} \{ I_m^c(r, \mu_0, \mu)\cos m(\phi - \phi_0) + I_m^s(r, \mu_0, \mu)\sin m(\phi - \phi_0) \}. \tag{O.14}$$

Combining eqn. O.13 and eqn. O.14 we find

$$\left\{ \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu} + \frac{\sqrt{1-\mu^2}\sqrt{1-\mu_0{}^2}}{r}\cos(\phi - \phi_0)\frac{\partial}{\partial\mu_0} \right\} I(r, \mu_0, \phi_0, \mu, \phi)$$

$$+ \frac{\sqrt{1-\mu^2}\sqrt{1-\mu_0{}^2}}{r}\frac{\mu_0}{1-\mu_0^2}\sin(\phi - \phi_0)$$

$$\cdot \sum_{m=0}^{\infty} \left\{ -m\, I_m^c(r,\mu_0,\mu) \sin m(\phi-\phi_0) + m\, I_m^s(r,\mu_0,\mu) \cos m(\phi-\phi_0) \right\}. \tag{O.15}$$

Since we are interested in the mean intensity

$$\begin{aligned} \overline{I}(r,\theta,\phi) &= \frac{1}{4\pi}\int_0^{2\pi} d\phi_0 \int_0^{\pi} \sin\theta d\theta_0\, I(r,\theta_0,\phi_0,\phi,\theta) \\ &= \frac{1}{4\pi}\int_0^{2\pi} d\phi_0 \int_{-1}^{1} d\mu_0\, I(r,\mu_0,\phi_0,\phi,\mu) \end{aligned} \tag{O.16}$$

we average eqn. O.16 over azimuth to get

$$\begin{aligned} &\mu \frac{\partial I_0^c(r,\mu_0,\mu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I_0^c(r,\mu_0,\mu)}{\partial \mu} + \frac{1}{2}\frac{\sqrt{1-\mu^2}\sqrt{1-\mu_0^2}}{r}\frac{\partial I_1^c(r,\mu_0,\mu)}{\partial \mu_0} \\ &+ \frac{1}{2}\frac{\sqrt{1-\mu^2}\sqrt{1-\mu_0^2}}{r}\frac{\mu_0}{1-\mu_0^2} I_1^c(r,\mu_0,\mu). \end{aligned} \tag{O.17}$$

Note that only the cosine terms ‘survived’ the averaging over azimuth.