Appendix P

Reciprocity, Duality and Effects of Surface Reflection

The purpose of this Appendix is to provide some details that were omitted in §6.10 regarding the relationship between the reflection and transmission for *unidirectional* (parallel beam or 'solar') and *uniform* (isotropic over the downward hemisphere) illumination of an inhomogeneous slab. The reflectance and transmittance for unidirectional illumination of a slab will be shown to be equivalent to the angular distribution of the azimuthally-averaged reflected and transmitted intensities, respectively, pertaining to uniform illumination of the slab with unit incident intensity. For an *inhomogeneous* slab the transmittance for unidirectional illumination from one side (e.g. the *top*) is equivalent to the angular distribution of the intensity pertaining to illumination from the other side (the *bottom*) of the slab. We will then derive an analytic expression for the intensity reflected from a Lambert surface underlying an inhomogeneeous slab, which in turn is required to derive simple analytic expressions for the reflectance and transmittance of an inhomogeneous slab overlying a partially reflecting surface in terms of the solution pertaining to the same slab overlying a *black* surface.

AP.1 Principle of Reciprocity

If the angular scattering depends only on the scattering angle, i. e. the angle between the direction of incidence and the direction in which the photon is scattered, then the phase function may be written

$$p(\Theta) = p(cos\Theta) = p[uu' + (1-u^2)^{\frac{1}{2}}(1-u'^2)^{\frac{1}{2}}cos(\phi - \phi')] \qquad (P.1)$$

where we have used eqn. 3.22. We see that the phase function satisfies the following relations

$$p(\mu,\phi;\mu',\phi') = p(\mu',\phi';\mu,\phi) \tag{P.2}$$

$$p(-\mu,\phi;-\mu',\phi') = p(\mu',\phi';\mu,\phi) \tag{P.3}$$

$$p(\mu,\phi;-\mu',\phi') = p(-\mu',\phi';\mu,\phi) = p(\mu',\phi';-\mu,\phi). \tag{P.4}$$

The above relations are usually referred to as Helmholtz' reciprocity principle. They are a consequence of time reversal invariance and they apply to a single scattering event.

AP.2 Homogeneous Slab

For a slab of finite thickness multiple scattering cannot, in general, be neglected. Therefore we do not expect reciprocity to be directly applicable. What is important here is, however, that the above reciprocity relations imply the following reciprocity rules for the reflectance and transmittance of a homogeneous slab of arbitrary (but finite) thickness τ^*

$$\rho(\tau^*;\mu,\phi;\mu_0,\phi_0) = \rho(\tau^*;\mu_0,\phi_0;\mu,\phi) \tag{P.5}$$

$$\mathcal{T}(\tau^*;\mu,\phi;\mu_0,\phi_0) = \mathcal{T}(\tau^*;\mu_0,\phi_0;\mu,\phi). \tag{P.6}$$

The radiation reflected and transmitted by the slab may be expressed as

$$I^+(0,\mu,\mu_0,\phi) = \mu_0 F^s \rho(\tau^*;\mu,\phi;\mu_0,\phi_0) \tag{P.7}$$

$$I^-(\tau^*,\mu,\mu_0,\phi) = \mu_0 F^s \mathcal{T}(\tau^*;\mu,\phi;\mu_0,\phi_0) \tag{P.8}$$

where $\mu_0 F^s$ is the (vertical) flux of the incident 'solar' radiation. Averaging over azimuth, we obtain

$$I^+(0,\mu,\mu_0) = \mu_0 F^s \rho(\tau^*;\mu;\mu_0) \tag{P.9}$$

$$I^-(\tau^*,\mu,\mu_0) = \mu_0 F^s \mathcal{T}(\tau^*;\mu;\mu_0) \tag{P.10}$$

where

$$\rho(\tau^*;\mu,\mu_0) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \rho(\tau^*;\mu,\phi;\mu_0,\phi_0) \tag{P.11}$$

$$\mathcal{T}(\tau^*;\mu,\mu_0) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \mathcal{T}(\tau^*;\mu,\phi;\mu_0,\phi_0). \tag{P.12}$$

AP.2.1 Collimated incidence

The reflectance and transmittance for collimated beam ('solar') incidence are obtained by integration

$$\rho^{beam}(\tau^*,\mu_0) = \frac{2\pi}{\mu_0 F^s}\int_0^1 d\mu\mu I^+(0,\mu,\mu_0)\mu d\mu = 2\pi\int_0^1 d\mu\mu\rho(\tau^*,\mu,\mu_0) \tag{P.13}$$

$$\mathcal{T}^{beam}(\tau^*,\mu_0) = \frac{2\pi}{\mu_0 F^s}\int_0^1 d\mu\mu I^-(\tau^*,\mu,\mu_0) = 2\pi\int_0^1 d\mu\mu\mathcal{T}(\tau^*,\mu,\mu_0). \tag{P.14}$$

Another integration yields the spherical albedo and transmittance

$$\bar{\rho}^{beam}(\tau^*) = 2\int_0^1 d\mu_0\mu_0\rho(\tau^*,\mu_0) = 4\pi\int_0^1 d\mu\mu\int_0^1 d\mu_0\rho(\tau^*,\mu,\mu_0) \tag{P.15}$$

$$\bar{\mathcal{T}}^{beam}(\tau^*) = 2\int_0^1 d\mu_0\mu_0\rho(\tau^*,\mu_0) = 4\pi\int_0^1 d\mu\mu\int_0^1 d\mu_0\mathcal{T}(\tau^*,\mu,\mu_0). \tag{P.16}$$

The superscript 'beam' is used to remind us that the illumination is collimated, the tilde (˜) sign that it is from below, and the overbar (¯) sign that we are are dealing with a *spherical* quantity.

AP.2.2 Uniform incidence

The angular distributions of the reflected and transmitted intensities for *uniform* illumination with unit incident intensity ($F^s = 1$) are

$$I^{+(uni)}(0,\mu) = 2\pi\int_0^1 d\mu_0 I^+(0,\mu,\mu_0) = 2\pi\int_0^1 d\mu_0\mu_0\rho(\tau^*,\mu,\mu_0) \tag{P.17}$$

$$I^{-(uni)}(\tau^*,\mu) = 2\pi\int_0^1 d\mu_0 I^-(\tau^*,\mu,\mu_0) = 2\pi\int_0^1 d\mu_0\mu_0\mathcal{T}(\tau^*,\mu,\mu_0). \tag{P.18}$$

The flux albedo and transmittance ($F^{-(uni)}(0) = \pi$) are given by

$$\frac{F^{+(uni)}(0)}{\pi} = 2\int_0^1 d\mu\mu I^+(0,\mu) = 4\pi\int_0^1 d\mu\mu\int_0^1 d\mu_0\rho(\tau^*,\mu,\mu_0) \tag{P.19}$$

$$\frac{F^{-(uni)}(\tau^*)}{\pi} = 2\int_0^1 d\mu\mu I^-(\tau^*,\mu) = 4\pi\int_0^1 d\mu\mu\int_0^1 d\mu_0\mathcal{T}(\tau^*,\mu,\mu_0). \tag{P.20}$$

The superscript 'uni' is used to remind us that the illumination is uniform.

AP.2.3 Duality

Since $\rho(\tau^*;\mu,\mu_0) = \rho(\tau^*;\mu_0,\mu)$ and $\mathcal{T}(\tau^*;\mu,\mu_0) = \mathcal{T}(\tau^*;\mu_0,\mu)$, the duality relations given in §6.10 follow by comparing the above expressions for collimated and uniform incidence.

AP.3 Inhomogeneous Slab

The expressions given above pertaining to a homogeneous slab will now be generalized to apply to a vertically inhomogeneous slab. We must distinguish between illumination from the top and the bottom. Thus, considering first illumination from the *top* we find that the same expressions as before (given by eqns. P.13–P.16 and eqns. P.17–P.20 above) apply for unidirectional and uniform illumination, respectively. However, for *unidirectional* and *uniform* illumination from the *bottom* we obtain the following expressions

$$\tilde{\rho}^{beam}(\tau^*,\mu_0) = 2\pi\int_0^1 d\mu\mu\tilde{\rho}(\tau^*,\mu,\mu_0);$$
$$\tilde{I}^{-(uni)}(\tau^*,\mu) = 2\pi\int_0^1 d\mu_0\mu_0\tilde{\rho}(\tau^*,\mu,\mu_0), \qquad \text{(P.21)}$$

$$\tilde{\mathcal{T}}^{beam}(\tau^*,\mu_0) = 2\pi\int_0^1 d\mu\mu\tilde{\mathcal{T}}(\tau^*,\mu,\mu_0);$$
$$\tilde{I}^{+(uni)}(0,\mu) = 2\pi\int_0^1 d\mu_0\mu_0\tilde{\mathcal{T}}(\tau^*,\mu,\mu_0), \qquad \text{(P.22)}$$

$$\tilde{\bar{\rho}}^{beam}(\tau^*) = 4\pi\int_0^1 d\mu\mu\int_0^1 d\mu_0\tilde{\rho}(\tau^*,\mu,\mu_0);$$
$$\frac{\tilde{F}^{-(uni)}(\tau^*)}{\pi} = 4\pi\int_0^1 d\mu\mu\int_0^1 d\mu_0\tilde{\rho}(\tau^*,\mu,\mu_0) \qquad \text{(P.23)}$$

$$\tilde{\bar{\mathcal{T}}}^{beam}(\tau^*) = 4\pi\int_0^1 d\mu\mu\int_0^1 d\mu_0\tilde{\mathcal{T}}(\tau^*,\mu,\mu_0);$$
$$\frac{\tilde{F}^{+(uni)}(0)}{\pi} = 4\pi\int_0^1 d\mu\mu\int_0^1 d\mu_0\tilde{\mathcal{T}}(\tau^*,\mu,\mu_0). \qquad \text{(P.24)}$$

The superscript ‘beam’ is used to remind us that the illumination is collimated, the tilde (˜) sign that it is from below, and the overbar (¯) sign that we are are dealing with a *spherical* quantity.

AP.3.1 Reciprocity and Duality

As noted in §6.10 for an inhomogeneous slab the reflectance and transmittance satisfy the following reciprocity relations

$$\begin{aligned}\rho(\tau^*,\mu,\mu_0) &= \rho(\tau^*,\mu_0,\mu); \qquad \tilde{\rho}(\tau^*,\mu,\mu_0) = \tilde{\rho}(\tau^*,\mu_0,\mu);\\ \mathcal{T}(\tau^*,\mu,\mu_0) &= \tilde{\mathcal{T}}(\tau^*,\mu_0,\mu). \end{aligned} \tag{P.25}$$

A crucial difference between the homogeneous and the *in*homogeneous slab is the reciprocity relating the transmittance due to illumination from one side to the illumination from the other side. Of course, for a homogeneous slab it makes no difference to which side we apply the illumination.

By comparing the expressions pertinent for collimated and uniform incidence and using these reciprocity relations we find that it is now a simple matter to generalize the duality relations for a homogeneous slab to obtain the expressions valid for an inhomogeneous slab provided in §6.10.

AP.4 Derivation of the Reflected Intensity Component I_r

In §6.11 we derived simple analytic expressions for the radiation reflected and transmitted by a slab overlying a partially reflecting (Lambert) surface in terms of the reflected intensity reflected at the lower boundary, I_r. In fact, the quantity

$$\frac{I_r}{\mu_0 F^s} = \frac{\rho_L \mathcal{T}(\mu_0; 2\pi)}{\pi(1-\bar{\tilde{\rho}}\rho_L)} = \frac{\rho_L \mathcal{T}(-\hat{\Omega}_0, -2\pi)}{\pi(1-\bar{\tilde{\rho}}\rho_L)}$$

appears in Eqs. 6.79 and 6.80 for the bidirectional reflectance and tranmittance of a slab overlying a Lambertian surface. Below we derive an expression for I_r in terms of the reflectance and transmittance pertinent to an inhomogeneous slab overlying a black (i.e *non-reflecting* surface).

In general, the intensity reflected at the lower boundary, $I^+(\tau^*,\mu,\phi)$, is related to the incident intensity, $I^-(\tau^*,\mu,\phi)$, through

$$I^+(\tau^*,\mu,\phi) = \int_0^{2\pi} d\phi' \int_0^1 d\mu'\mu'\rho(-\mu',\phi';\mu,\phi) I^-(\tau^*;\mu',\phi') \tag{P.26}$$

or by averaging over azimuth

$$I^+(\tau^*,\mu) = 2\pi \int_0^1 d\mu'\mu'\rho(-\mu',\mu) I^-(\tau^*,\mu') \equiv I_r \tag{P.27}$$

where $\rho(-\mu', \phi'; \mu, \phi)$ is the bidirectional reflectance of the surface, and $\rho(-\mu', \mu)$ its azimuthal mean. Here I_r is a constant because we are dealing with a Lambert surface for which $\rho(-\mu', \mu) = \rho_L = constant$.

Next we consider the total reflected intensity $I^+_{tot}(0; \mu, \phi)$, which consists of three separate components: (a) the contribution from the atmosphere assuming a non-reflecting or black lower boundary ($\rho = 0$); (b) the diffusely-transmitted component arising from I_r (see eqn. 5.30); and (c) the directly-transmitted component arising from I_r. In mathematical terms, we write

$$I^+_{tot}(0; \mu, \phi) = I^+(0; \mu, \phi; \rho = 0) + \int_0^{2\pi} d\phi' \int_0^1 d\mu' \mu' \tilde{\mathcal{T}}_d(\mu', \phi'; \mu, \phi) I_r + I_r e^{-\tau^*/\mu} \tag{P.28}$$

since we have assumed that the reflected intensity is azimuth-independent and given by eqn. P.27. Removing I_r from the integral (it is independent of angle), we combine terms (b) and (c), to obtain the total transmittance

$$I_r \Big[e^{-\tau^*/\mu} + \int_0^{2\pi} d\phi' \int_0^1 d\mu' \mu' \tilde{\mathcal{T}}_d(\mu'; \mu) \Big]. \tag{P.29}$$

The second term is recognized as the diffuse part of the *hemispherical-directional transmittance* $\tilde{\mathcal{T}}_d(2\pi; \mu)$ pertaining to radiation incident from below. We note the absence of azimuthal dependence. As shown for the flux reflectance in Problem 3.1(b), reciprocity also applies to the *flux transmittance*

$$\tilde{\mathcal{T}}_d(2\pi; \mu) = \tilde{\mathcal{T}}_d(\mu; 2\pi). \tag{P.30}$$

In words, the *hemispherical-directional transmittance* is also the *directional-hemispherical transmittance.* Therefore

$$I^+_{tot}(0; \mu, \phi) = I^+(0; \mu, \phi; \rho = 0) + I_r \tilde{\mathcal{T}}(\mu; 2\pi) \tag{P.31}$$

where $\tilde{\mathcal{T}}(\mu; 2\pi) = e^{-\tau^*/\mu} + \tilde{\mathcal{T}}_d(\mu; 2\pi)$, or the total transmittance is the sum of the beam and diffuse transmittances. We note that the remaining ϕ-dependence of the total intensity is due to the first term, and is traceable to a ϕ-dependence of the collimated beam illumination. The extra term in the above equation (arising from the boundary) is azimuthally independent by assumption (Lambert reflector).

The first term may be expressed in terms of the incident radiation field (assumed to be a collimated solar beam) and the atmospheric reflectance as $\mu_0 F^s \rho(-\mu_0, \phi_0; \mu, \phi)$. Therefore

$$I^+_{tot}(0; \mu, \phi) = \mu_0 F^s \rho(-\mu_0, \phi_0; \mu, \phi) + I_r \tilde{\mathcal{T}}(\mu; 2\pi). \tag{P.32}$$

Proceeding in a similar manner we find that the transmitted intensity can be expressed as

$$I_{tot}^{-}(\tau^*;\mu,\phi) = \mu_o F^s \mathcal{T}(-\mu_0,\phi_0;\mu,\phi) + I_r \tilde{\rho}(\mu;2\pi). \qquad \text{(P.33)}$$

Here the first term is the diffusely transmitted intensity, while the second term stems from radiation reflected first from the surface and then from the atmosphere above.

It remains to determine I_r. Setting the reflected flux πI_r equal to a constant, ρ_L, times the downward flux at τ^*, we have

$$\pi I_r = \rho_L \left[\mu_0 F^s e^{-\tau^*/\mu_0} + \mu_0 F^s \mathcal{T}_d(\mu_0;2\pi) + \pi I_r \tilde{\tilde{\rho}}\right]. \qquad \text{(P.34)}$$

The first term on the left side is the directly-transmitted solar flux, the second term is the diffusely-transmitted component for a completely black surface, and the third term is the (downward) reflected component due to the upward reflection from the Lambert surface followed by downward reflection by the atmosphere. We recognize $\tilde{\tilde{\rho}}$ as the *spherical albedo* pertaining to illumination from below. Solving the above for I_r we obtain

$$I_r = \frac{\mu_0 F^s \rho_L \left[e^{-\tau^*/\mu_0} + \mathcal{T}_d(\mu_0;2\pi)\right]}{\pi(1-\tilde{\tilde{\rho}}\rho_L)} = \frac{\mu_0 F^s \rho_L \mathcal{T}(\mu_0;2\pi)}{\pi(1-\tilde{\tilde{\rho}}\rho_L)} \qquad \text{(P.35)}$$

where we have once again combined the sum of the direct and diffuse transmittances into a total transmittance.